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The standard model and its generalizations in the Epstein–Glaser approach to renormalization theory

D R Grigore

Department of Theoretical Physics, Institute of Atomic Physics, Bucharest–Măgurele, PO Box MG 6, România

E-mail: grigore@theor1.theory.nipne.ro and grigore@theory.nipne.ro

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Abstract. We continue our study of non-Abelian gauge theories in the framework of the Epstein–Glaser approach to renormalization theory. We consider the case when massive spin-1 bosons are present in the theory and we modify appropriately the analysis of the origin of the gauge invariance performed in a preceding paper in the case of null-mass spin-1 bosons. Then we are able to extend a result of Dütsch and Scharf concerning the uniqueness of the standard model, consistent with renormalization theory. In fact we consider the most general case, i.e. the consistent interaction of r spin-1 bosons, and we do not impose any restrictions on the gauge group and the mass spectrum of the theory. We show that, besides the natural emergence of a group structure (as in the massless case), we obtain new conditions of a group theoretical nature, namely the existence of a certain representation of the gauge group associated to the Higgs fields. Some other mass relations connecting the structure constants of the gauge group and the masses of the bosons emerge naturally. The proof is carried out using the Epstein–Glaser approach to renormalization theory.

1. Introduction

The traditional approach to renormalization theory starts from Bogoliubov axioms imposed on the S-matrix (or equivalently on the chronological products) and translates them into axioms on the Feynman amplitudes (the so-called Hepp axioms). Then one tries to find explicit solutions of these axioms using some regularization procedure and extracting, in a consistent way, the ultraviolet infinities. Even for a real scalar field this task is not very easy, but the theory becomes extremely complicated when one considers systems with gauge invariance. Rigorous analysis performed by Becchi, Rouet and Stora shows the tremendous complexity of the theory. In recent years, a new way to consider renormalization theory was advocated by Professor G Scharf starting from the analysis of Epstein and Glaser [13]. In this approach one also starts from Bogoliubov axioms on the chronological products, but one tries to find solutions in a purely recursive way using the support properties of various distributions appearing in the problem and a procedure called distribution splitting. More importantly, in this approach one can shed new light on the problem of gauge invariance: one can argue that a consistent interaction (in the sense of perturbation theory) involving spin-1 bosons should be a gauge invariant interaction.

In a preceding paper [16] we extended a result of Aste and Scharf [1] concerning the uniqueness of the non-Abelian gauge theory describing the consistent interaction of r null-mass bosons of spin-1. We showed that the gauge invariance principle is a natural consequence of the description of spin-1 particles in a factor Hilbert space. Here, one considers an auxiliary Hilbert

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space $\mathcal{H}^{gh}$ (describing physical and ghost particles) and assumes the existence of a supercharge Q operator acting in it. Then gauge invariance expresses the possibility of factorizing the S-matrix to the physical space, which is usually constructed according to the cohomological-type formula $\mathcal{H}_{phys} = \text{Ker}(Q)/\text{Im}\,(Q)$. The obstructions to such a factorization process are the well known anomalies. The main problem in this approach is the fact that this factorization can be implemented only in the adiabatic limit, so one has to solve simultaneously the ultraviolet and the infrared problems. If the spin-1 bosons are massless, then one cannot solve this combined problem. The case when the spin-1 bosons of non-null mass are admitted into the problem was studied by Aste, Dütsch and Scharf [3, 11] for the concrete case of the electroweak interaction, i.e. when the gauge group is exactly $SU(2) \times U(1)$. In this paper we analyse the same problem when considering that the spin-1 bosons can have non-null masses and we do not impose any restriction on their number and masses (also we do not take into account matter fields). We will show that in this case one can hope for a theory where the ultraviolet problems are completely under control and the infrared problems seem to be less severe.

We should also mention that, as in [16], we fill a gap in the existing literature concerning the legitimacy of using the identification $\mathcal{H}_{phys} = \text{Ker}(Q)/\text{Im}\,(Q)$. Indeed, from the physical point of view, one should proceed in strict accordance with the dogma of the second quantization of Fock–Cook as follows. One starts with a one-particle Hilbert space H which is usually some representation of the Poincaré group; in our case we consider a massive spin-1 particle. Then one chooses the statistics, which in this case should be Bose–Einstein, and considers as a physical space the associated symmetric Fock space $\mathcal{F}^+(\mathsf{H})$. It is not obvious that this coincides with $\mathcal{H}_{phys} = \text{Ker}(Q)/\text{Im}\,(Q)$, although it is usually asserted that this follows from the analysis of Kugo and Ojima [25]. In this paper, as in [16] for massless spin-1 bosons, we prove that the identification of the two Hilbert spaces is a rigorous mathematical fact. As a byproduct, we have a simpler analysis of the unitarity of the S-matrix.

Summing up, the main objective of this paper is to show that the very construction of the standard model follows from Bogoliubov axioms and the factorization condition to the physical Hilbert space $\mathcal{H}_{phys}$; these conditions determine in a rather strict way the structure of the interaction Lagrangian for (massive) bosons of spin-1. To make the connection with the Feynman graph terminology, it will follow from the proof that only tree diagrams of the first and second order of the perturbation theory are considered. The remarkable fact about this approach is the fact that no classical Lagrangian, subject afterwards to some quantization procedure, is needed.

We should also mention an apparent drawback of the approach, which is in fact common to all known approaches to gauge theories. To construct the $\mathcal{H}^{gh}$ one has to consider some explicit representations for the one-particle Hilbert space (for instance, to work with the Wigner representation or the helicity representation for the irreducible projective unitary representations of the Poincaré group). Also one has to add a set of ghost fields such that the cohomology of the supercharge gives the physical Fock space. These two choices are non-canonical, i.e. no formulae independent of the representation are available. Of course, it is plausible to conjecture that for two distinct choices one gets the same physics, i.e. there is a unitary transformation intertwining them. This conjecture is suggested by the analysis of gauge theories in the framework of classical field theory where the ghost fields are some canonical objects associated to a principal fibre bundle. In some particular cases, the conjecture can actually be proved. For instance, one can consider some special choice for the Hilbert space and the ghost fields corresponding to the so-called linear gauges. In these gauges the S-matrix *a priori* depends on a gauge parameter and one can prove that this dependence drops out after factorization to the physical Hilbert space [4].

We mention that since the first version of this paper was posted in the hep-th archive (hep-th/9810078) two other papers on the same subject have appeared [12, 27]. They are mainly concerned with the case when all Bosons are massive and the adiabatic limit exists. The price to pay is the introduction of some supplementary scalar fields.

The structure of the paper is as follows. In the next section we generalize the description of non-null mass spin-1 bosons on similar lines as in [16]. A modification of the supercharge will be necessary. As in [11] and [3], the appearance of the Higgs fields seems unavoidable. In section 3 we construct the first-order S-matrix following closely the lines of the computations from [16] to which we will frequently refer. We will also be able to give a generic form for the second-order S-matrix. Next we impose gauge invariance for the second-order perturbation theory and obtain, as expected, that the structure is rather tight, i.e. there are severe restrictions on the various coefficients of the various Wick monomials entering the interaction Lagrangian. Moreover we naturally obtain that some of these coefficients can be organized into a representation of dimension r of the gauge group, which is nothing more than the representation T of the Higgs fields. In particular, some very complicated computations from [11], leading to the cancellation of possible anomalies, are nothing more than the representation property of T. Some interesting mass relations connecting the structure constants, the representation T and the masses of the bosons naturally emerge. These relations seems to be new in the literature, at least to our knowledge, and they have the merit of having a rigorous status.

Finally, we test the generic formalism on the standard model of electroweak interactions. In this way the results of [3] are re-obtained. In the last section we list some future problems which have to be solved.

Other papers also treating the quantization of massive bosons of spin-1 in the Epstein–Glaser approach are [5–7, 18–20, 22] and [23].

Concerning the relationship between the Epstein–Glaser–Scharf approach and the traditional ways of computing various effects in the standard model (see, for instance, [24]) we believe that no serious discrepancies should appear. We are basing this assessment on the fact that the starting point is the same: the Bogoliubov axioms of perturbative renormalization theory. On the other hand, one should note that the expression of the BRST transformation in this approach is a linearized version of the usual one. This does not mean that the results should be different from the usual approach but this point is not completely settled and deserves further investigations. We should also mention a recent paper of [21] on the quantum Noether method which can probably be used to derive the same results as ours and, moreover, to prove equivalence with the results of the traditional approach.

2. Spin-1 relativistic free particles with positive mass

2.1. General description

As in [16], we take the one-particle space of the problem H to be the Hilbert space of a unitary irreducible representation of the Poincaré group. We give below the relevant formulae for particles of mass $m > 0$ and spin-1.

The upper hyperboloid of mass $m \geqslant 0$ is by definition the set of functions $X_m^+ \equiv \{p \in \mathbb{R}^4 | \|p\|^2 = m^2,\ p_0 > 0\}$ which are square integrable with respect to the Lorentz invariant measure $\mathrm{d}\alpha_m^+(p) \equiv \frac{\mathrm{d}\boldsymbol{p}}{2\omega(\boldsymbol{p})}$ (in fact only classes of functions identical up to null-measure sets are considered). The conventions are the following: $\|\cdot\|$ is the Minkowski norm defined by $\|p\|^2 \equiv p \cdot p$ and $p \cdot q$ is the Minkowski bilinear form: $p \cdot q \equiv p_0 q_0 - \boldsymbol{p} \cdot \boldsymbol{q}$.

Let us consider the Hilbert space $\mathsf{H} \equiv L^2(X_m^+, \mathbb{C}^4, d\alpha_m^+)$ with the scalar product

$$\langle \phi, \psi \rangle \equiv \int_{X_m^+} d\alpha_m^+(p) \qquad \langle \phi(p), \psi(p) \rangle_{\mathbb{C}^4} \tag{2.1.1}$$

where $\langle u, v \rangle_{\mathbb{C}^4} \equiv \sum_{i=1}^4 \overline{u_i} v_i$ is the usual scalar product from $\mathbb{C}^4$. In this Hilbert space we have the following (non-unitary) representation of the Poincaré group:

$$(U_{a,\Lambda}\phi)(p) \equiv e^{ia\cdot p} \Lambda \cdot \phi(\Lambda^{-1} \cdot p) \qquad \text{for} \quad \Lambda \in \mathcal{L}^\uparrow \qquad (U_{I_t}\phi)(p) \equiv \overline{\phi(I_s \cdot p)}. \tag{2.1.2}$$

We define on H the following non-degenerate sesquilinear form:

$$(\phi, \psi) \equiv \int_{X_m^+} d\alpha_m^+(p) \qquad g^{\mu\nu} \overline{\phi_\mu(p)} \psi_\nu(p) \tag{2.1.3}$$

where the indices μ, ν take the values 0, 1, 2, 3 and the summation convention over the dummy indices is used. This form behaves naturally with respect to the representation (2.1.2).

Now we immediately have the following.

Lemma 2.1. *Let us consider the following subspace of* H*:*

$$\mathsf{H}_m \equiv \{\phi \in \mathsf{H} | p^\mu \phi_\mu(p) = 0\}. \tag{2.1.4}$$

Then the sesquilinear form $(\cdot, \cdot)|_{\mathsf{H}_m}$ *is strictly positively defined.*

As a consequence we have the following.

Proposition 2.2. *The representation (2.1.2) of the Poincaré group leaves invariant the subspace* H_m *and the restriction of this representation to this subspace (also denoted by* U*) is equivalent to the unitary irreducible representation* $\mathsf{H}^{[m,1]}$ *of the Poincaré group (describing particles of mass* $m > 0$ *and spin-1 [28]).*

By definition, the couple (H_m, U) is called a *spin-1 boson* of mass m.

We turn now to the second quantization procedure applied to such an elementary system. We express the (bosonic) *Fock space* of the system

$$\mathcal{F}_m \equiv \mathcal{F}^+(H_m) \equiv \oplus_{n\geqslant 0} \mathcal{H}'_n \qquad \mathcal{H}'_0 \equiv \mathbb{C} \tag{2.1.5}$$

as a subspace of an auxiliary Fock space

$$\mathcal{H} \equiv \mathcal{F}^+(H) \equiv \oplus_{n\geqslant 0} \mathcal{H}_n \qquad \mathcal{H}_0 \equiv \mathbb{C}. \tag{2.1.6}$$

One canonically identifies the nth-particle subspace $\mathcal{H}_n$ with the set of Borel functions $\Phi^{(n)}_{\mu_1,\ldots,\mu_n}\colon (X_m^+)^{\times n} \to \mathbb{C}$ which are square summable:

$$\int_{(X_m^+)^{\times n}} \prod_{i=1}^n d\alpha_m^+(k_i) \sum_{\mu_1,\ldots,\mu_n=0}^3 |\Phi^{(n)}_{\mu_1,\ldots,\mu_n}(k_1, \ldots, k_n)|^2 < \infty \tag{2.1.7}$$

and verify the symmetry property

$$\Phi^{(n)}_{\mu_{P(1)},\ldots,\mu_{P(n)}}(k_{P(1)}, \ldots, k_{P(n)}) = \Phi^{(n)}_{\mu_1,\ldots,\mu_n}(k_1, \ldots, k_n) \qquad \forall P \in \mathcal{P}_n. \tag{2.1.8}$$

In $\mathcal{H}$ one has natural extensions of the expression of the scalar product (2.1.1) and of the sesquilinear form (2.1.3). We also have a (non-unitary) representation of the Poincaré group given by the well known formula $\mathcal{U}_g \equiv \Gamma(U_g), \forall g \in \mathcal{P}$; here U_g is given by (2.1.2).

Now one has the following from lemma 2.1.

Lemma 2.3. *Let us consider the following subspace of* $\mathcal{H}$*:*

$$\mathcal{H}' \equiv \mathcal{F}^{+}(\mathsf{H}_m) = \oplus_{n\geqslant 0}\mathcal{H}'_n. \tag{2.1.9}$$

Then $\mathcal{H}'_n$*,* $n \geqslant 1$ *is generated by elements of the form* $\phi_1 \vee \cdots \vee \phi_n$*,* $\phi_1, \ldots, \phi_n \in \mathsf{H}_m$ *and, in the representation adopted previously for the Hilbert space* $\mathcal{H}_n$*, we can take*

$$\mathcal{H}'_n = \{\Phi^{(n)} \in \mathcal{H}_n | k_1^{\nu_1}\Phi^{(n)}_{\nu_1,\ldots,\nu_n}(k_1,\ldots,k_n) = 0\}. \tag{2.1.10}$$

Moreover, the sesquilinear form $(\cdot,\cdot)|_{\mathcal{H}'}$ *is strictly positively defined.*

Finally we have the following.

Proposition 2.4. *There exists a canonical isomorphism of Hilbert spaces*

$$\mathcal{F}_m \simeq \mathcal{H}'. \tag{2.1.11}$$

Now we can define the corresponding field as an operator on the Hilbert space $\mathcal{H}$ in complete analogy to the electromagnetic field: we define for every $p \in X_m^{+}$ the annihilation and creation operators

$$(A_\nu(p)\Phi)^{(n)}_{\mu_1,\ldots,\mu_n}(k_1,\ldots,k_n) \equiv \sqrt{n+1}\,\Phi^{(n+1)}_{\nu,\mu_1,\ldots,\mu_n}(p,k_1,\ldots,k_n) \tag{2.1.12}$$

and

$$(A_\nu^{\dagger}(p)\Phi)^{(n)}_{\mu_1,\ldots,\mu_n}(k_1,\ldots,k_n) \equiv -2\omega(\boldsymbol{p})\frac{1}{\sqrt{n}} \times \sum_{i=1}^{n}\delta(\boldsymbol{p}-\boldsymbol{k_i})g_{\nu\mu_i}\Phi^{(n-1)}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_n}(k_1,\ldots,\hat{k}_i,\ldots,k_n). \tag{2.1.13}$$

Then one has a list of properties which are formally identical to the corresponding one from the null-mass case. First we have the *canonical commutation relations* (CAR)

$$\begin{aligned}&[A_\nu(p), A_\rho(p')] = 0 \qquad [A_\nu^{\dagger}(p), A_\rho^{\dagger}(p')] = 0\\ &[A_\nu(p), A_\rho^{\dagger}(p')] = -2\omega(\boldsymbol{p})g_{\nu\rho}\delta(\boldsymbol{p}-\boldsymbol{p}')\mathbf{1}\end{aligned} \tag{2.1.14}$$

and the relation

$$(A_\nu^{\dagger}(p)\Psi, \Phi) = (\Psi, A_\nu(p)\Phi) \qquad \forall\Psi,\Phi \in \mathcal{H} \tag{2.1.15}$$

which shows that $A_\nu^{\dagger}(p)$ is the adjoint of $A_\nu(p)$ with respect to the sesquilinear form $(\cdot,\cdot)$.

Next we have a natural behaviour with respect to the action of the Poincaré group:

$$\begin{aligned}&\mathcal{U}_{a,\Lambda}A_\nu(p)\mathcal{U}_{a,\Lambda}^{-1} = \mathrm{e}^{\mathrm{i}a\cdot p}(\Lambda^{-1})_\nu{}^{\rho}A_\rho(\Lambda\cdot p) \qquad \forall\Lambda \in \mathcal{L}^{\uparrow}\\ &\mathcal{U}_{I_t}A_\nu(p)\mathcal{U}_{I_t}^{-1} = (I_t)_\nu{}^{\rho}A_\rho(I_s\cdot p)\end{aligned} \tag{2.1.16}$$

and a similar relation for $A_\nu^{\dagger}(p)$.

Finally, we define the *field operators in the point* x according to

$$A_\nu(x) \equiv A_\nu^{(+)}(x) + A_\nu^{(-)}(x) \tag{2.1.17}$$

where the expressions appearing in the right-hand side are the positive (negative) frequency parts and are defined by

$$\begin{aligned}A_\nu^{(+)}(x) &\equiv \frac{1}{(2\pi)^{3/2}}\int_{X_m^{+}}\mathrm{d}\alpha_m^{+}(p)\mathrm{e}^{\mathrm{i}p\cdot x}A_\nu^{\dagger}(p)\\ A_\nu^{(-)}(x) &\equiv \frac{1}{(2\pi)^{3/2}}\int_{X_m^{+}}\mathrm{d}\alpha_m^{+}(p)\mathrm{e}^{-\mathrm{i}p\cdot x}A_\nu(p).\end{aligned} \tag{2.1.18}$$

The properties of the field operators $A_\nu(x)$ are contained in the following elementary proposition.

Proposition 2.5. *The following relations are true:*

$$(A_\nu(x)\Psi, \Phi) = (\Psi, A_\nu(x)\Phi) \qquad \forall\Psi, \Phi \in \mathcal{H} \tag{2.1.19}$$

$$(\Box + m^2)A_\nu(x) = 0 \tag{2.1.20}$$

and

$$[A_\mu^{(\mp)}(x), A_\nu^{(\pm)}(y)] = -g_{\mu\nu} D_m^{(\pm)}(x-y) \times \mathbf{1} \qquad [A_\mu^{(\pm)}(x), A_\nu^{(\pm)}(y)] = 0. \tag{2.1.21}$$

As a consequence we also have

$$[A_\mu(x), A_\nu(y)] = -g_{\mu\nu} D_m(x-y) \times \mathbf{1}, \tag{2.1.22}$$

here

$$D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x) \tag{2.1.23}$$

is the Pauli–Jordan distribution and $D_m^{(\pm)}(x)$ are given by

$$D_m^{(\pm)}(x) \equiv \pm \frac{1}{(2\pi)^{3/2}} \int_{X_m^+} \mathrm{d}\alpha_m^+(p) \mathrm{e}^{\mp \mathrm{i} p\cdot x}. \tag{2.1.24}$$

Let us note that we have

$$(\Box + m^2) D_m^{(\pm)}(x) = 0 \qquad (\Box + m^2) D_m(x) = 0. \tag{2.1.25}$$

We turn now to the constructions of observables on the Fock space of the spin-1 boson $\mathcal{F}_m \simeq \mathcal{H}'$ by self-adjoint operators on the Hilbert space $\mathcal{H}$. If O is such an operator on $\mathcal{H}'$ then it induces naturally an operator (also denoted by O) on $\mathcal{H}$ which leaves invariant the subspace $\mathcal{H}'$. These types of observables on $\mathcal{H}$ are called *gauge invariant observables*.

The description of possible interactions between the spin-1 field and matter follows the same ideas. Let us consider that the (Fock) space of the 'matter' fields is denoted by $\mathcal{H}_{\text{matter}}$. Then, in the hypothesis of weak coupling, one can argue that the Hilbert space of the combined system is $\mathcal{H}_{\text{total}} \equiv \mathcal{F}_m \otimes \mathcal{H}_{\text{matter}}$. It is easy to see that, if we define $\tilde{\mathcal{H}} \equiv \mathcal{H} \otimes \mathcal{H}_{\text{matter}}$ and $\tilde{\mathcal{H}}' \equiv \mathcal{H}' \otimes \mathcal{H}_{\text{matter}}$, we have as before

$$\mathcal{H}_{\text{total}} \simeq \tilde{\mathcal{H}}'. \tag{2.1.26}$$

In the Hilbert space $\tilde{\mathcal{H}}$ we can define as usual the expressions for the spin-1 field and all properties listed previously stay true. Typical interaction terms have the form

$$T_1(x) \equiv A_\nu(x) j^\nu(x) \tag{2.1.27}$$

where $j^\nu(x)$ are some Wick polynomials in the 'matter' fields called *currents*. Then conservation of the current is a sufficient and necessary condition such that the expression (2.1.27) induces, in the adiabatic limit, a well defined expression on the Hilbert space $\mathcal{H}_{\text{total}}$.

For higher-order chronological products, one can establish a similar expression:

$$T_n(x_1, \ldots, x_n) \equiv \sum_{k=0}^{n} : A_{\nu_1}(x_1) \ldots A_{\nu_k}(x_k) : j^{\nu_1,\ldots,\nu_k}(x_1, \ldots, x_n) \tag{2.1.28}$$

and the condition of factorization, in the adiabatic limit, amounts again to the conservation of the multi-currents $j^{\nu_1,\ldots,\nu_k}(x_1, \ldots, x_n)$. The conservation of these multi-currents can be heuristically connected with the gauge invariance of the S-matrix (see [26] Ch 4.6).

2.2. Quantization with ghost fields

In this subsection we give an alternative description of the Fock space $\mathcal{F}_m$ using the ghost fields following rather closely the arguments from [16]. However, in the case of positive mass particles it seems that it is not sufficient to introduce the Fermionic ghosts and one has also to introduce a bosonic ghost.

In [16], the Hilbert space was constructed by acting on the vacuum state with the electromagnetic potentials A_ν and the pair of ghost fields of null mass and Fermi statistics u and $\tilde{u}$. In the case of spin-1 bosons of mass $m > 0$, we generate $\mathcal{H}^{\text{gh}}$ by acting on the vacuum with the potentials A_ν and the triplet of ghost fields of the same mass u, $\tilde{u}$ and Φ, such that the first two are Fermionic and the last one is a bosonic field. We will need some explicit representation for $\mathcal{H}^{\text{gh}}$. Taking into account the general structure outlined above, we should have

$$\mathcal{H}^{\text{gh}} = \oplus_{n,w,l,s=0}^{\infty}\mathcal{H}_{nwls} \tag{2.2.1}$$

where one can identify $\mathcal{H}_{nwls}$ with the set of Borel functions $\Phi^{(nwls)}_{\mu_1,\ldots,\mu_n}\colon\ (X_0^+)^{n+w+l+s} \to \mathbb{C}$ which are square integrable with respect to the product measure $(\alpha_m^+)^{\times(n+w+l+s)}$

$$\sum_{n,w,l,s=0}^{\infty}\int_{(X_m^+)^{n+w+l+s}} \mathrm{d}\alpha_m^+\,(K)\,\mathrm{d}\alpha_m^+\,(P)\,\mathrm{d}\alpha_m^+\,(Q)\,\mathrm{d}\alpha_m^+\,(R)$$
$$\times \sum_{\mu_1,\ldots,\mu_n=0}^{3} |\Phi^{(nwls)}_{\mu_1,\ldots,\mu_n}(K;P;Q;R)|^2 \leqslant \infty \tag{2.2.2}$$

(here $K \equiv (k_1,\ldots,k_n)$, $P \equiv (p_1,\ldots,p_w)$, $Q \equiv (q_1,\ldots,q_l)$ and $R \equiv (r_1,\ldots,r_l)$), verifying the symmetry property

$$\begin{aligned}\Phi^{(nwls)}_{\mu_{P(1)},\ldots,\mu_{P(n)}}&(k_{P(1)},\ldots,k_{P(n)};p_{Q(1)},\ldots,p_{Q(w)};q_{R(1)},\ldots,q_{R(l)};r_{T(1)},\ldots,r_{T(s)})\\ &= (-1)^{|Q|+|R|}\Phi^{(nwls)}_{\mu_1,\ldots,\mu_n}(k_1,\ldots,k_n;p_1,\ldots,p_w;q_1,\ldots,q_l;r_1,\ldots,r_s),\\ &\forall P\in\mathcal{P}_n\qquad Q\in\mathcal{P}_w\qquad R\in\mathcal{P}_l\qquad T\in\mathcal{P}_s.\end{aligned} \tag{2.2.3}$$

In this representation we can construct the following annihilation operators:

$$(A_\nu(t)\Phi)^{(nwls)}_{\mu_1,\ldots,\mu_n}(k_1,\ldots,k_n;P;Q;R) = \Phi^{(n+1,wls)}_{\nu,\mu_1,\ldots,\mu_n}(t,k_1,\ldots,k_n;P;Q;R) \tag{2.2.4}$$

$$(b(t)\Phi)^{(nwls)}_{\mu_1,\ldots,\mu_n}(K;p_1,\ldots,p_w;Q;R) = \Phi^{(n,w+1,ls)}_{\mu_1,\ldots,\mu_n}(K;t,p_1,\ldots,p_w;Q;R) \tag{2.2.5}$$

$$(c(t)\Phi)^{(nwls)}_{\mu_1,\ldots,\mu_n}(K;P;q_1,\ldots,q_l;R) = (-1)^w\sqrt{l+1}\,\Phi^{(nw,l+1,s)}_{\mu_1,\ldots,\mu_n}(K;P;t,q_1,\ldots,q_l;R) \tag{2.2.6}$$

and

$$(a(t)\Phi)^{(nwls)}_{\mu_1,\ldots,\mu_n}(K;P;Q;r_1,\ldots,r_s) = \sqrt{s+1}\,\Phi^{(nwl,s+1)}_{\mu_1,\ldots,\mu_n}(K;P;Q;t,r_1,\ldots,r_s). \tag{2.2.7}$$

Similar expressions can be written for the creation operators. These verify usual canonical (anti)commutation relations and behave naturally with respect to the Poincaré transform.

Then the fields

$$u(x) \equiv \frac{1}{(2\pi)^{3/2}}\int_{X_m^+}\mathrm{d}\alpha_m^+\,(q)[\mathrm{e}^{-\mathrm{i}q\cdot x}b(q) + \mathrm{e}^{\mathrm{i}q\cdot x}c^*(q)] \tag{2.2.8}$$

$$\tilde{u}(x) \equiv \frac{1}{(2\pi)^{3/2}}\int_{X_m^+}\mathrm{d}\alpha_m^+\,(q)[-\mathrm{e}^{-\mathrm{i}q\cdot x}c(q) + \mathrm{e}^{\mathrm{i}q\cdot x}b^*(q)] \tag{2.2.9}$$

and

$$\Phi(x) \equiv \frac{1}{(2\pi)^{3/2}}\int_{X_m^+}\mathrm{d}\alpha_m^+\,(q)[\mathrm{e}^{-\mathrm{i}q\cdot x}a(q) + \mathrm{e}^{\mathrm{i}q\cdot x}a^*(q)] \tag{2.2.10}$$

are called Fermionic (resp. bosonic) ghost fields.

These verify the wave equations

$$(\Box + m^2)u(x) = 0 \qquad (\Box + m^2)\tilde{u}(x) = 0 \qquad (\Box + m^2)\Phi(x) = 0 \quad (2.2.11)$$

and we have the usual canonical (anti)commutation relations

$$\{u(x), \tilde{u}(y)\} = D_m(x-y)\mathbf{1} \qquad [\Phi(x), \Phi(y)] = D_m(x-y)\mathbf{1} \qquad (2.2.12)$$

and all other (anti)commutators are null. Now we can define the operator

$$Q \equiv \int_{X_m^+} \mathrm{d}\alpha_m^+(q)[k^\mu(A_\mu(k)c^*(k) + A_\mu^\dagger(k)b(k)) + \mathrm{i}m(a(k)c^*(k) - a^*(k)b(k))] \qquad (2.2.13)$$

called *supercharge*. Its properties are summarized in the following proposition which can be proved by elementary computations. By Φ_0 we denote the vacuum state.

Proposition 2.6. *The following relations are valid:*

$$Q\Phi_0 = 0 \qquad (2.2.14)$$

$$\begin{aligned}&[Q, A_\mu^\dagger(k)] = -k_\mu c^*(k) \qquad \{Q, b^*(k)\} = k^\mu A_\mu^\dagger(k) - \mathrm{i}ma^*(k)\\ &\{Q, c^*(k)\} = 0 \qquad [Q, a^*(k)] = \mathrm{i}mc^*(k)\end{aligned} \qquad (2.2.15)$$

$$\begin{aligned}&[Q, A_\mu(k)] = k_\mu b(k) \qquad \{Q, b(k)\} = 0\\ &\{Q, c(k)\} = k^\mu A_\mu(k) + \mathrm{i}ma(k) \qquad [Q, a(k)] = \mathrm{i}mb(k)\end{aligned} \qquad (2.2.16)$$

$$\begin{aligned}&\{Q, u(x)\} = 0 \qquad \{Q, \tilde{u}(x)\} = -\mathrm{i}(\partial^\mu A_\mu(x) + m\Phi(x))\\ &[Q, A_\mu(x)] = \mathrm{i}\partial_\mu u(x) \qquad [Q, \Phi(x)] = \mathrm{i}mu(x)\end{aligned} \qquad (2.2.17)$$

$$Q^2 = 0 \qquad (2.2.18)$$

$$\mathrm{Im}\,(Q) \subset \mathrm{Ker}(Q) \qquad (2.2.19)$$

and

$$\mathcal{U}_g Q = Q\mathcal{U}_g \qquad \forall g \in \mathcal{P}. \qquad (2.2.20)$$

Moreover, one can express the supercharge in terms of the ghost fields as follows:

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3x\, (\partial^\mu A_\mu(x) + m\Phi(x)) \overset{\leftrightarrow}{\partial_0} u(x). \qquad (2.2.21)$$

(The succession of the preceding formulae suggests the most convenient way to derive them; for instance, from (2.2.16) and (2.2.15) one derives that $\{Q, Q\} = 0$ and gets (2.2.18). In particular (2.2.18) justifies the terminology of the supercharge and (2.2.19) indicates that it might be interesting to take the quotient. Indeed, we will rigorously prove that this quotient coincides with $\mathcal{F}_m$.)

We can give the explicit expression of the supercharge in this representation. Starting from the definition (2.2.13) we immediately get

$$\begin{aligned}(Q\Phi)^{(nwls)}_{\mu_1,\ldots,\mu_n}(K; P; Q; R) &= (-1)^w \sqrt{\frac{n+1}{l}}\\ &\times \sum_{i=1}^{l} (-1)^{i-1} q_i^\nu \Phi^{(n+1,w,l-1,s)}_{\nu,\mu_1,\ldots,\mu_n}(q_i, K; P; q_1, \ldots, \hat{q}_i, \ldots, q_l; R)\\ &- \sqrt{\frac{w+1}{n}} \sum_{i=1}^{n} (k_i)_{\mu_i} \Phi^{(n-1,w+1,l,s)}_{\mu_1,\ldots,\hat{\mu}_i,\ldots,\mu_n}(k_1, \ldots, \hat{k}_i, \ldots, k_n; k_i, P; Q; R)\\ &+ \mathrm{i}m(-1)^w \sqrt{\frac{s+1}{l}} \sum_{i=1}^{l} (-1)^{i-1} \Phi^{(n,w,l-1,s+1)}_{\mu_1,\ldots,\mu_n}(K; P; q_1, \ldots, \hat{q}_i, \ldots, q_l; q_i, R)\end{aligned}$$

$$-\mathrm{i}m\sqrt{\frac{w+1}{s}}\sum_{i=1}^{s}\Phi^{(n,w+1,l,s-1)}_{\mu_1,\ldots,\mu_n}(K;r_i,P;Q;r_1,\ldots,\hat{r}_i,\ldots,r_s) \tag{2.2.22}$$

where, of course, we use Bourbaki convention $\sum_{\emptyset} \equiv 0$.

Now we introduce on $\mathcal{H}^{\mathrm{gh}}$ a *Krein operator* according to

$$(J\Phi)^{(nwls)}(K;P;Q;R) \equiv (-1)^{wl}(-g)^{\otimes n}\Phi^{(nlws)}(K;Q;P;R). \tag{2.2.23}$$

The properties of this operator are contained in the following proposition.

Proposition 2.7. *The following relations are verified:*

$$J^* = J^{-1} = J \tag{2.2.24}$$

$$\begin{aligned} &Jb(p)J = c(p) \qquad Jc(p)J = b(p) \\ &JA^*_\mu(p)J = A^\dagger_\mu(p) \qquad Ja(p)J = a(p) \end{aligned} \tag{2.2.25}$$

$$JQJ = Q^* \tag{2.2.26}$$

and

$$\mathcal{U}_g J = J\mathcal{U}_g \qquad \forall g \in \mathcal{P}. \tag{2.2.27}$$

Here O^* is the adjoint of the operator O with respect to the scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}^{\mathrm{gh}}$.

We define, as usual, a sesquilinear form on $\mathcal{H}^{\mathrm{gh}}$ according to

$$(\Psi,\Phi) \equiv \langle\Psi, J\Phi\rangle. \tag{2.2.28}$$

Then this form is non-degenerate. It is convenient to denote the conjugate of the arbitrary operator O with respect to the sesquilinear form $(\cdot,\cdot)$ by $O^\dagger$, i.e.

$$(O^\dagger\Psi,\Phi) = (\Psi, O\Phi). \tag{2.2.29}$$

Then the following formula is available:

$$O^\dagger = JO^*J. \tag{2.2.30}$$

As a consequence, we have

$$A_\mu(x)^\dagger = A_\mu(x) \qquad u(x)^\dagger = u(x) \qquad \tilde{u}(x)^\dagger = -\tilde{u}(x) \qquad \Phi(x)^\dagger = \Phi(x). \tag{2.2.31}$$

From (2.2.27) it follows that we have

$$(\mathcal{U}_g\Psi,\mathcal{U}_g\Phi) = (\Psi,\Phi) \qquad \forall g \in \mathcal{P}^\uparrow \qquad (\mathcal{U}_{I_t}\Psi,\mathcal{U}_{I_t}\Phi) = \overline{(\Psi,\Phi)}. \tag{2.2.32}$$

As in [16], we give a description of the factor space $\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)$. We will construct a 'homotopy' for the supercharge Q.

Proposition 2.8. *Let us define the operator*

$$\tilde{Q} \equiv -\frac{1}{2m^2}\int_{X_m^+} \mathrm{d}\alpha_m^+(q)[k^\mu(A_\mu(k)b^*(k) + A^\dagger_\mu(k)c(k)) + \mathrm{i}m(a^*(k)c(k) - a(k)b^*(k))]. \tag{2.2.33}$$

Then the following relation is valid:

$$Y \equiv \{Q,\tilde{Q}\} = N_a + N_b + N_c + X \tag{2.2.34}$$

where N_a (N_c, N_c) are particle number operators for the ghosts of type a (resp. b, c) and

$$X \equiv -\frac{1}{m^2}\int_{X_m^+} \mathrm{d}\alpha_m^+(k)k^\mu k^\nu A^\dagger_\mu(k)A_\nu(k). \tag{2.2.35}$$

Moreover the following relations are true:

$$\tilde{Q}^2 = 0 \tag{2.2.36}$$

and

$$[Y,Q] = 0 \qquad [Y,\tilde{Q}] = 0. \tag{2.2.37}$$

The operator $\tilde{Q}$ it is called the *homotopy* of Q. The operator Y is not invertible, but as in [16] we have the following.

Proposition 2.9. *The operator $Y|_{\mathcal{H}_{nwls}}$ is invertible iff $w + l + s > 0$.*

Proof. An alternative expression for the operator X defined by (2.2.35) is

$$X = A \otimes \mathbf{1} \tag{2.2.38}$$

where the operator A acts only on the bosonic variables and is given by the expression

$$A = \mathrm{d}\Gamma\,(P). \tag{2.2.39}$$

Here $\mathrm{d}\Gamma$ is the familiar Cook functor defined by

$$\mathrm{d}\Gamma\,(P)\psi_1 \otimes \cdots \otimes \psi_n \equiv P\psi_1 \otimes \psi_2 \cdots \otimes \psi_n + \cdots \psi_1 \otimes \psi_2 \cdots \otimes P\psi_n \tag{2.2.40}$$

and the operator P is, in our case, given by

$$(P\psi)_\mu(k) \equiv \frac{1}{m^2} k_\mu k^\nu \psi^\nu(k). \tag{2.2.41}$$

We immediately obtain that P is a projector, i.e. $P^2 = P$, and we have, as in the case of massless bosons of spin-1, the direct sum decomposition of the one-particle bosonic subspace into the direct sum of $\mathrm{Ran}(P)$ and $\mathrm{Ran}(1 - P)$. Let us consider a basis in the one-particle bosonic subspace formed by a basis $f_i, i \in \mathbb{N}$ of $\mathrm{Ran}(P)$ and a basis $g_i, i \in \mathbb{N}$ of $\mathrm{Ran}(1 - P)$. A basis in the nth-particle bosonic subspace is of the form

$$f_{i_1} \vee \cdots f_{i_r} \vee g_{j_1} \vee \cdots \vee g_{j_t} \qquad r, t \in \mathbb{N} \qquad r + t = n.$$

Applying the operator A to such a vector gives the same vector multiplied by r. So, in the basis chosen above, the operator A is diagonal with diagonal elements from $\mathbb{N}$. It follows that the operator $Y|_{\mathcal{H}_{nwls}}$ can also be exhibited into a diagonal form with diagonal elements of the form $w + l + s + r$, $r \in \mathbb{N}$. It is obvious that for $w + l + s > 0$ this is an invertible operator. □

Accordingly, we have the following corollary.

Corollary 2.10. *Let us define $\mathcal{H}_0 \equiv \oplus_{n\geqslant 0}\mathcal{H}_{n000}$ and $\mathcal{H}_1 \equiv \oplus_{n\geqslant 0, w+l+s>0}\mathcal{H}_{nwls}$. Then the operator Y has the block-diagonal form*

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_0 \end{pmatrix} \tag{2.2.42}$$

with Y_1 an invertible operator.

We now have the following fundamental result.

Proposition 2.11. *There exists the following vector space isomorphism:*

$$\mathrm{Ker}(Q)/\mathrm{Im}\,(Q) \simeq \mathcal{H}' \tag{2.2.43}$$

where the subspace $\mathcal{H}'$ has been defined in the previous subsection (see the lemmas 2.3).

Proof. (i) As in [16] one can prove that if $\Phi \in \mathrm{Ker}(Q)$ then we have the decomposition

$$\Phi = Q\psi + \tilde{\Phi} \tag{2.2.44}$$

where

$$\tilde{\Phi}^{(nwls)} = 0 \qquad w + l + s > 0. \tag{2.2.45}$$

The condition $Q\Phi = 0$ amounts now to $Q\tilde{\Phi} = 0$ or, with the explicit expression of the supercharge (2.2.22),

$$q^{\nu}\tilde{\Phi}^{(n+1,0,0,0)}_{\nu,\mu_1,\ldots,\mu_n}(q, k_1, \ldots, k_n; \emptyset; \emptyset; \emptyset) = 0 \qquad \forall n \in \mathbb{N} \tag{2.2.46}$$

i.e. the ensemble $\{\tilde{\Phi}^{(n00)}\}|_{n\in\mathbb{N}}$ is an element from $\mathcal{H}'$ (see lemma 2.3).

It remains to see in what conditions such $\tilde{\Phi}$ is an element from $\mathrm{Im}\,(Q)$, i.e. we have $\tilde{\Phi} = Q\chi$. It is clear that only the components $\chi^{(n100)}$ should be taken non-null. Then the expression of the supercharge (2.2.22) gives the following condition:

$$(Q\chi)^{(n001)}_{\mu_1,\ldots,\mu_n}(K; \emptyset; \emptyset; r) = -\mathrm{i}m\chi^{(n100)}_{\mu_1,\ldots,\mu_n}(K; r; \emptyset; \emptyset) = 0. \tag{2.2.47}$$

Because the mass m of the boson is non-null, we get $\chi = 0 \quad \Rightarrow \tilde{\Phi} = 0$ and we obtain the assertion from the statement. □

We finally get the following theorem as in [16].

Theorem 2.12. *The isomorphism (2.2.43) extends to a Hilbert space isomorphism*

$$\overline{\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)} \simeq \mathcal{F}_m$$

and the factorized representation of the Poincaré group coincides with the representation acting into the space $\mathcal{H}'$.

We close with an important observation. One can easily see that one can take the limit $m \searrow 0$ in the expressions for the various Hilbert spaces and quantum fields and also on the expression of the supercharge Q. (The expression $\tilde{Q}$ does not have the limit in the obvious way, but this is not very important, because this expression had played only an auxiliary role.) In this limit we can write

$$\mathcal{H}^{\mathrm{gh}} \simeq \mathcal{H}^{\mathrm{gh}}_0 \otimes \mathcal{H}_{\Phi} \tag{2.2.48}$$

where $\mathcal{H}^{\mathrm{gh}}_0$ is the Hilbert space generated by the fields $A_\mu(x)$, $u(x)$, $\tilde{u}(x)$ and $\mathcal{H}_\Phi$ is generated by the scalar ghosts. Then the supercharge (2.2.13) takes the form

$$Q = Q' \otimes \mathbf{1} \tag{2.2.49}$$

where Q' coincides formally with the expression of Q for $m \searrow 0$ but acts only in $\mathcal{H}^{\mathrm{gh}}_0$. Moreover, we have

$$\overline{\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)} \simeq \overline{\mathrm{Ker}(Q')/\mathrm{Im}\,(Q')} \otimes \mathcal{H}_\Phi \tag{2.2.50}$$

i.e. we can see that the states from $\mathcal{H}_\Phi$ decouple completely and can be considered physical. Moreover, one can see that, in this case, nothing prevents us from considering that the scalar 'ghost' has a non-zero mass. This observation is essential for the construction of the standard model, because a scalar 'ghost' field corresponding to a null-mass boson, if considered a physical field of non-zero mass, is nothing more than the Higgs field [3]. Let us stress that this observation has to be raised to the status of a postulate; it agrees with the construction of the standard model as we shall see in the following section and brings no mathematical inconsistencies.

2.3. *Gauge-invariant observables*

As in [16], we denote by $\mathcal{W}$ the linear space of all Wick monomials on the Fock space $\mathcal{H}^{\mathrm{gh}}$ i.e. containing the fields $A_\mu(x)$, $u(x)$, $\tilde{u}(x)$ and $\Phi(x)$. If M is such a Wick monomial, we define by $\mathrm{gh}_\pm(M)$ the degree in $\tilde{u}$ (resp. in u). The *ghost number* is, by definition, the expression

$$\mathrm{gh}(M) \equiv \mathrm{gh}_+(M) - \mathrm{gh}_-(M) \tag{2.3.1}$$

i.e. we conserve the same expression as in the massless case. The BRST operator also has the same expression; it is given by

$$d_Q M \equiv: QM : -(-1)^{\mathrm{gh}(M)} : MQ : \tag{2.3.2}$$

on monomials M and can be extended by linearity to the whole $\mathcal{W}$.

Most of the formulae from [16] stay true:

$$d_Q^2 = 0 \tag{2.3.3}$$

$$\begin{aligned} d_Q u &= 0 \qquad d_Q \tilde{u} = -\mathrm{i}(\partial^\mu A_\mu + m\Phi) \\ d_Q A_\mu &= \mathrm{i}\partial_\mu u \qquad d_Q \Phi = \mathrm{i}mu \end{aligned} \tag{2.3.4}$$

$$d_Q(MN) = (d_Q M)N + (-1)^{\mathrm{gh}(M)} M(d_Q N) \qquad \forall M, N \in \mathcal{W}. \tag{2.3.5}$$

The class of *all* observables on the factor space emerges (see theorem 2.12): an operator $O : \mathcal{H}^{\mathrm{gh}} \to \mathcal{H}^{\mathrm{gh}}$ induces a well defined operator $[O]$ on the factor space $\overline{\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)} \simeq \mathcal{F}_m$ if and only if it verifies

$$d_Q O|_{\mathrm{Ker}(Q)} = 0. \tag{2.3.6}$$

Not all operators verifying the condition (2.3.6) are interesting. In fact, the operators of the type $d_Q O$ induce a null operator on the factor space; explicitly, we have

$$[d_Q O] = 0. \tag{2.3.7}$$

Moreover, in this case, the following formula is true for the matrix elements of the factorized operator $[O]$:

$$([\Psi], [O][\Phi]) = (\Psi, O\Phi). \tag{2.3.8}$$

If the interaction Lagrangian is a Wick monomial $T_1 \in \mathcal{W}$ with $\mathrm{gh}(T_1) \neq 0$ then the S-matrix is trivial.

The analysis of the possible interactions between the bosonic spin-1 field and 'matter' follows the usual lines (see [16]). Let $\mathcal{H}_{\mathrm{matter}}$ be the corresponding Hilbert space of the matter fields. It is elementary to see that we can realize the total Hilbert space $\mathcal{H}_{\mathrm{total}} \equiv \mathcal{F}_m \otimes \mathcal{H}_{\mathrm{matter}}$ as the factor space $\mathrm{Ker}(Q)/\mathrm{Im}\,(Q)$ where the supercharge Q is defined on $\tilde{\mathcal{H}}_{\mathrm{gh}} \equiv \mathcal{H}_{\mathrm{gh}} \otimes \mathcal{H}_{\mathrm{matter}}$ by the obvious substitution $Q \to Q \otimes \mathbf{1}$.

We define on $\tilde{\mathcal{H}}_{\mathrm{gh}}$ the interaction Lagrangian of the same form (2.1.27) where the current $j^\mu(x)$ is a Wick polynomial and it is conserved.

3. Massive Yang–Mills fields

3.1. The general setting

As in [16], we first define in an unambiguous way what we mean by Yang–Mills fields. The main modification is that now all the fields will carry an additional index $a = 1, \ldots, r$ and this can be realized with an appropriate modification of the Hilbert spaces (auxiliary or physical). So we have the fields $A_{a\mu}, u_a, \tilde{u}_a, \Phi_a$ $a = 1, \ldots, r$ given by the following expressions:

$$A_{a\mu}(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X^+_{m_a}} \mathrm{d}\alpha^+_{m_a}(p)[\mathrm{e}^{-\mathrm{i}p\cdot x} A_{a\mu}(p) + \mathrm{e}^{\mathrm{i}p\cdot x} A^\dagger_{a\mu}(p)] \tag{3.1.1}$$

$$u_a(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X^+_{m_a}} \mathrm{d}\alpha^+_{m_a}(q)[\mathrm{e}^{-\mathrm{i}q\cdot x} b_a(q) + \mathrm{e}^{\mathrm{i}q\cdot x} c^\dagger_a(q)] \tag{3.1.2}$$

$$\tilde{u}_a(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X^+_{m_a}} \mathrm{d}\alpha^+_{m_a}(q)[-\mathrm{e}^{-\mathrm{i}q\cdot x} c_a(q) + \mathrm{e}^{\mathrm{i}q\cdot x} b^\dagger_a(q)] \tag{3.1.3}$$

and

$$\Phi_a(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X^+_{m_a}} \mathrm{d}\alpha^+_{m_a}(q)[\mathrm{e}^{-\mathrm{i}q\cdot x} a_a(q) + \mathrm{e}^{\mathrm{i}q\cdot x} a_a^\dagger(q)]. \tag{3.1.4}$$

As in [16], this amounts to considering that the one-particle subspace is a direct sum of r copies of elementary heavy bosons of masses m_a, $a = 1, \ldots, r$ and spin-1.

These fields verify the following equations of motion:

$$\begin{aligned} &(\square + m_a^2)u_a(x) = 0 \qquad (\square + m_a^2)\tilde{u}_a(x) = 0 \\ &(\square + m_a^2)\Phi_a(x) = 0 \qquad a = 1, \ldots, r. \end{aligned} \tag{3.1.5}$$

The canonical (anti)commutation relations are

$$\begin{aligned} &[A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab} g_{\mu\nu} D_{m_a}(x-y) \times \mathbf{1} \\ &\{u_a(x), \tilde{u}_b(y)\} = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1} \\ &[\Phi_a(x), \Phi_b(y)] = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1} \end{aligned} \tag{3.1.6}$$

and all other (anti)commutators are null. The supercharge is given by (see (2.2.13))

$$Q \equiv \sum_{a=1}^{r} \int_{X^+_{m_a}} \mathrm{d}\alpha^+_{m_a}(q)[k^\mu(A_{a\mu}(k)c_a^*(k) + A^\dagger_{a\mu}(k)b_a(k)) + \mathrm{i}m_a(a_a(k)c_a^\dagger(k) - a_a^*(k)b_a(k))] \tag{3.1.7}$$

and verifies all the expected properties.

The Krein operator has an expression similar to (2.2.23) and can be used to construct a sesquilinear form as in (2.2.28). Then relations of the type (2.2.31) are still true:

$$\begin{aligned} &A_{a\mu}(x)^\dagger = A_{a\mu}(x) \qquad u_a(x)^\dagger = u_a(x) \\ &\tilde{u}_a(x)^\dagger = -\tilde{u}_a(x) \qquad \Phi_a(x)^\dagger = \Phi_a(x). \end{aligned} \tag{3.1.8}$$

As a consequence, proposition 2.11 and the main theorem 2.12 stay true.

The ghost degree is defined in an obvious way and the expression for the BRST operator (2.3.2) is the same in this more general framework and the corresponding properties are easy to obtain. In particular we have (see (2.3.4))

$$\begin{aligned} &\mathrm{d}_Q u_a = 0 \qquad \mathrm{d}_Q \tilde{u}_a = -\mathrm{i}(\partial_\mu A_a^\mu + m_a \Phi_a) \\ &\mathrm{d}_Q A_a^\mu = \mathrm{i}\partial^\mu u_a \qquad \mathrm{d}_Q \Phi_a = \mathrm{i} m_a u_a \qquad \forall a = 1, \ldots, r. \end{aligned} \tag{3.1.9}$$

We close this section with a general remark. If we take into account the last observation from the preceding subsection, it appears that it is possible to make some of the masses null in the formalism presented above. In this case the corresponding scalar ghosts can be considered as physical fields and they will be called Higgs fields.

Moreover, we do not have to assume that they are massless, i.e. if some boson field A_a^μ has zero mass $m_a = 0$, we can suppose that the corresponding Higgs field Φ_a has a non-zero mass: m_a^H. Of course, if the mass of the vector field A_a^μ is non-zero $m_a \neq 0$, then we have $m_a^H = m_a$. It is extremely convenient to define the expression m_a^* to be equal to m_a if $m_a \neq 0$ and to m_a^H if $m_a = 0$. Then in the last relation (3.1.6) one must make $m_a \to m_a^*$. Moreover, this process of attributing a non-zero mass to the scalar partners of the zero-mass vector fields should not influence the BRST transformation formula (3.1.9); that is, this formula remains unchanged. We raise this comment to the status of a postulate, as at the end of the preceding subsection, and we make the following comments. Suppose that we have s zero-mass spin-1 bosons and $r - s$ non-null spin-1 bosons in our theory; here $s \leqslant r$ is arbitrary. The consistency of the formalism requires that the scalar partners of every one of the non-null spin-1 bosons should be a fictitious particle—a ghost—with exactly the same mass. On the contrary, the scalar partners of the s

zero-mass spin-1 bosons can be taken as *physical* particles of arbitrary mass. This postulate is in agreement with the case of the standard model, where we have exactly one zero-mass boson—the photon—and exactly one physical scalar partner—the Higgs particle. Moreover, this postulate does not bring mathematical inconsistencies. However, we should mention that this postulate has its limitations: for instance it is possible that in the case when all bosons are massive there are no solutions in this framework. In other words, models like the Higgs–Kibble model (see [25]) are not covered by this postulate. Recently [27], Professor Scharf noticed that one can consider a consistent theory with $t > s$ physical scalars and in this framework Higgs–Kibble-type models can be described, although their physical relevance is not as clear at the moment (see also [12]).

We will construct a perturbation theory *á la* Epstein–Glaser for the *free* fields A_a^μ, u_a, $\tilde{u}_a$ and Φ_a, $a = 1, \ldots, r$ in the auxiliary Hilbert space $\mathcal{H}_{\rm YM}^{{\rm gh},r}$ imposing the usual axioms of causality, unitarity and relativistic invariance. Moreover, we want the result to factorize to the physical Hilbert space in the adiabatic limit. This amounts to

$$\lim_{\epsilon\searrow 0} {\rm d}_Q \int_{(\mathbb{R}^4)^{\times n}} {\rm d}x_1 \ldots {\rm d}x_n g_\epsilon(x_1) \ldots g_\epsilon(x_n) T_n(x_1, \ldots, x_n)\Big|_{{\rm Ker}(Q)} = 0 \qquad \forall n \geqslant 1. \tag{3.1.10}$$

If this condition is fulfilled, then the chronological and the antichronological products do factorize to the physical Hilbert space and they give a perturbation theory verifying causality, unitarity and relativistic invariance.

We have to make a rather delicate comment on this point. One can argue that there are reasons to believe that the infrared (or adiabatic) limit does not exist, so the preceding relation does not have a rigorous status. Moreover, in the adiabatic limit, tri-linear Wick monomials are zero and this affects the argument of the next theorem. This seems to jeopardize the very nice physical interpretation of this relation, namely as a consistency condition (the factorization to the physical space of the S-matrix), which avoids the necessity to consider gauge invariance as a separate postulate. This problem is avoided by the Zürich group as follows [7–10]. One admits instead of (3.1.10) an 'infinitesimal' version, namely

$$ {\rm d}_Q T_n(x_1, \ldots, x_n) = {\rm i} \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T_{n/l}^\mu(x_1, \ldots, x_n) \tag{3.1.11}$$

where $T_{n/l}^\mu(x_1, \ldots, x_n)$ are some auxilliary chronological products which should be determined recursively, together with the standard chronological products. This implies that, instead of (3.1.10), we have

$$ {\rm d}_Q \int_{(\mathbb{R}^4)^{\times n}} {\rm d}x_1 \ldots {\rm d}x_n g_\epsilon(x_1) \ldots g_\epsilon(x_n) T_n(x_1, \ldots, x_n)\Big|_{{\rm Ker}(Q)} = O(\epsilon) \qquad \forall n \geqslant 1 \tag{3.1.12}$$

i.e. the factorization is valid only up to terms of order ϵ.

One should not touch the adiabatic limit, i.e. one should construct the matrix $S(g)$ for $g \not\equiv 1$. We should mention here that the basis for this new postulate is in fact that the relation (3.1.10) is considered as a heuristic idea. So the two relations are rather closely related and there is no severe opposition between them. But in this way, one will construct an object $S(g)$ which does not factorize to the physical space ${\rm Ker}(Q)/{\rm Im}\,(Q)$, as can easily be seen. This raises doubts about its physical interpretation.

Another way out would be to restrict the physical space $\mathcal{F}_m$ even further, that is to consider that only some of the states of this space are physically accessible, for instance only those states containing 'soft photons', as in the usual treatment of the infrared divergences. Then one would have to modify the factorization condition (3.1.10) (replacing ${\rm Ker}(Q)$ by a smaller subspace) and check that this does not invalidate some of the arguments from the next subsection, for example the linear independence arguments.

So, an honest point of view is the following one: the adiabatic problem is still open and it is presumably the main obstacle left for the construction of a complete rigorous version of the standard model. We conjecture that a nice 'cure' to this problem can be found and, in that case, we will be able to accept the consistency condition (3.1.10) as a rigorous mathematical fact. In this case gauge invariance will not be an independent postulate, as we advocate here. Until then, if we want to be completely rigorous, we are forced to replace (3.1.10) by (3.1.11) and deal afterwards with the adiabatic limit in the usual and rather unsatisfactory way.

3.2. The derivation of the Yang–Mills Lagrangian; first-order gauge invariance

In this subsection we completely exploit the condition of gauge invariance in the first-order perturbation theory, obtaining the generic form of the Yang–Mills interaction of spin-1 bosons. We assume the summation convention of the dummy indices $a, b, \ldots$.

Theorem 3.1. *Let us consider the operator*

$$T_1(g) = \int_{\mathbb{R}^4} \mathrm{d}x\, g(x) T_1(x) \tag{3.2.1}$$

defined on $\mathcal{H}^{\mathrm{gh},r}_{\mathrm{YM}}$ *with* T_1*, a Lorentz-invariant Wick polynomial in* A_μ*,* u*,* $\tilde{u}$ *and* Φ *verifying also* $\omega(T_1) \leqslant 4$*. If* $T_1(g)$ *induces a well defined non-trivial S-matrix, in the adiabatic limit, then it necessarily has the following form:*

$$T_1(g) = \int_{\mathbb{R}^4} \mathrm{d}x\, g(x)[T_{11}(x) + T_{12}(x) + T_{13}(x) + T_{14}(x) + T_{15}(x) + T_{16}(x)] \tag{3.2.2}$$

where we have introduced the following notations:

$$T_{11}(x) \equiv f_{abc}[:A_{a\mu}(x) A_{b\nu}(x) \partial^\nu A^\mu_c(x): - :A^\mu_a(x) u_b(x) \partial_\mu \tilde{u}_c(x):] \tag{3.2.3}$$

$$\begin{aligned} T_{12}(x) \equiv f'_{abc}[&:\Phi_a(x)\partial_\mu \Phi_b(x) A^\mu_c(x): - m_b :\Phi_a(x) A_{b\mu}(x) A^\mu_c(x): \\ &- m_b :\Phi_a(x) \tilde{u}_b(x) u_c(x):] \end{aligned} \tag{3.2.4}$$

$$T_{13}(x) \equiv f''_{abc} :\Phi_a(x)\Phi_b(x)\Phi_c(x): \tag{3.2.5}$$

$$T_{14}(x) \equiv g_{abcd} :\Phi_a(x)\Phi_b(x)\Phi_c(x)\Phi_d(x): \tag{3.2.6}$$

$$T_{15} \equiv h_{ab}[:A_{a\mu}(x) A^\mu_b(x): - 2 :\tilde{u}_a(x) u_b(x):] \qquad T_{16}(x) \equiv h'_{ab} :\Phi_a(x)\Phi_b(x):. \tag{3.2.7}$$

Here the various constants from the preceding expression are constrained by the following conditions:

- *the expressions* f_{abc} *are completely antisymmetric*

$$f_{abc} = -f_{bac} = -f_{acb} \tag{3.2.8}$$

  and verify

$$(m_a - m_b) f_{abc} = 0 \qquad \text{iff} \quad m_c = 0 \qquad \forall a, b = 1, \ldots, r; \tag{3.2.9}$$

- *the expressions* f'_{abc} *are antisymmetric in the indices a and b:*

$$f'_{abc} = -f'_{bac} \tag{3.2.10}$$

  and verify the relation

$$(m^H_a - m^H_b) f'_{abc} = 0 \qquad \text{iff} \quad m_a = m_b = m_c = 0 \qquad \forall a, b = 1, \ldots, r \tag{3.2.11}$$

  and are connected to f_{abc} *by*

$$f_{abc} m_c = f'_{cab} m_a - f'_{cba} m_b \qquad \forall a, b, c = 1, \ldots, r; \tag{3.2.12}$$

- *the expressions f''_{abc} are completely symmetric in all indices and remain undetermined for $m_a = m_b = m_c = 0$; for the opposite case they are given by*

$$f''_{abc} = \frac{1}{6m_c} f'_{abc}[(m_a^*)^2 - (m_b^*)^2 - m_a^2 + m_b^2] \qquad \text{iff} \quad m_c \neq 0 \qquad a, b = 1, \ldots, r; \tag{3.2.13}$$

- *the expressions g_{abcd} are non-zero if and only if $m_a = m_b = m_c = m_d = 0$ and they are completely symmetric;*
- *the expressions h_{ab} are symmetric*

$$h_{ab} = h_{ba} \tag{3.2.14}$$

and verify the relation

$$(m_a - m_b)h_{ab} = 0 \qquad \forall a, b = 1, \ldots, r; \tag{3.2.15}$$

- *the constants h'_{ab} are undetermined for $m_a = m_b = 0$ and in the opposite case are given by*

$$h'_{ab} = \frac{m_a}{2m_b} h_{ab} \qquad \text{iff} \quad m_b \neq 0 \qquad \forall a = 1, \ldots, r. \tag{3.2.16}$$

(We note that it is implicit in relations like (3.2.9), (3.2.11), etc that the summation convention over the dummy indices does not apply.)

Proof. (i) We follow closely the line of argument of theorem 4.1 from [16]. If we take into account Lorentz invariance, the power counting condition from the statement and the restriction of non-triviality $gh(T_1) = 0$, the list of linearly independent Wick monomials from [16] (formula 4.2.4 from section 4.2) is enlarged by new possibilities containing, of course, the scalar ghosts

- of degree 2:

$$\begin{aligned} T^{(1)'} &= h^{(1)}_{ab} : A_{a\mu}(x) A^\mu_b(x) : \\ T^{(2)'} &= h^{(2)}_{ab} : \tilde{u}_a(x) u_b(x) : \\ T^{(3)'} &= h^{(3)}_{ab} : \Phi_a(x) \Phi_b(x) : \end{aligned} \tag{3.2.17}$$

- of degree 3:

$$\begin{aligned} T^{(1)''} &= h^{(1)}_{abc} : \Phi_a(x) A_{b\mu}(x) A^\mu_c(x) : \\ T^{(2)''} &= h^{(2)}_{abc} : \Phi_a(x) \tilde{u}_b(x) u_c(x) : \\ T^{(3)''} &= h^{(3)}_{abc} : \Phi_a(x) \Phi_b(x) \Phi_c(x) : \\ T^{(4)''} &= h^{(4)}_{ab} : \Phi_a(x) \partial_\mu A^\mu_b(x) : \\ T^{(5)''} &= h^{(5)}_{ab} : \partial_\mu \Phi_a(x) A^\mu_b(x) : \end{aligned} \tag{3.2.18}$$

- of degree 4:

$$
\begin{aligned}
&T^{(1)} = f^{(1)}_{abc} : A_{a\mu}(x) A_{b\nu}(x) \partial^\nu A^\mu_c(x) : \\
&T^{(2)} = f^{(2)}_{abc} : A^\mu_a(x) u_b(x) \partial_\mu \tilde{u}_c(x) : \\
&T^{(3)} = f^{(3)}_{abc} : A^\mu_a(x) \partial_\mu u_b(x) \tilde{u}_c(x) : \\
&T^{(4)} = f^{(4)}_{abc} : \partial_\mu A^\mu_a(x) u_b(x) \tilde{u}_c(x) : \\
&T^{(5)} = f^{(5)}_{abc} : A_{a\mu}(x) A^\mu_b(x) \partial_\nu A^\nu_c(x) : \\
&T^{(6)} = g^{(1)}_{abcd} : A_{a\mu}(x) A^\mu_b(x) A_{c\nu}(x) A^\nu_d(x) : \\
&T^{(7)} = g^{(2)}_{abcd} : A_{a\mu}(x) A^\mu_b(x) u_c(x) \tilde{u}_d(x) : \\
&T^{(8)} = g^{(3)}_{abcd} : u_a(x) u_b(x) \tilde{u}_c(x) \tilde{u}_d(x) : \\
&T^{(9)} = g^{(4)}_{abcd} \varepsilon_{\mu\nu\rho\sigma} : A^\mu_a(x) A^\nu_b(x) A^\rho_c(x) A^\sigma_d(x) : \\
&T^{(10)} = g^{(1)}_{ab} : \partial_\mu A_{a\nu}(x) \partial^\mu A^\nu_b(x) : \\
&T^{(11)} = g^{(2)}_{ab} : \partial_\mu A^\mu_a(x) \partial_\nu A^\nu_b(x) : \\
&T^{(12)} = g^{(3)}_{ab} : \partial_\mu A_{a\nu}(x) \partial^\nu A^\mu_b(x) : \\
&T^{(13)} = g^{(4)}_{ab} : A^\mu_a(x) \partial_\mu \partial_\nu A^\nu_b(x) : \\
&T^{(14)} = g^{(5)}_{ab} \varepsilon_{\mu\nu\rho\sigma} : F^{\mu\nu}_a(x) F^{\rho\sigma}_b(x) : \\
&T^{(15)} = g^{(6)}_{ab} : \partial_\mu u_a(x) \partial^\mu \tilde{u}_b(x) : \\
&T^{(16)} = f^{(6)}_{abc} : \Phi_a(x) \Phi_b(x) \partial_\mu A^\mu_c(x) : \\
&T^{(17)} = f^{(7)}_{abc} : \Phi_a(x) \partial_\mu \Phi_b(x) A^\mu_c(x) : \\
&T^{(18)} = g^{(5)}_{abcd} : \Phi_a(x) \Phi_b(x) A_{c\mu}(x) A^\mu_d(x) : \\
&T^{(19)} = g^{(6)}_{abcd} : \Phi_a(x) \Phi_b(x) \tilde{u}_c(x) u_d(x) : \\
&T^{(20)} = g^{(7)}_{abcd} : \Phi_a(x) \Phi_b(x) \Phi_c(x) \Phi_d(x) : \\
&T^{(21)} = h^{(5)}_{ab} : \partial_\mu \Phi_a(x) \partial^\mu \Phi_b(x) : .
\end{aligned}
\tag{3.2.19}
$$

Without losing generality we can impose the following symmetry restrictions on the constants from the preceding list:

$$
\begin{array}{lll}
h^{(1)}_{ab} = h^{(1)}_{ba} & g^{(1)}_{abcd} = g^{(1)}_{bacd} = g^{(1)}_{abdc} = g^{(1)}_{cdab} & g^{(2)}_{abcd} = g^{(2)}_{bacd} \\
h^{(3)}_{ab} = h^{(3)}_{ba} \quad h^{(5)}_{ab} = h^{(5)}_{ba} & h^{(1)}_{abc} = h^{(1)}_{acb} & \\
g^{(3)}_{abcd} = -g^{(3)}_{bacd} = -g^{(3)}_{abdc} & g^{(i)}_{ab} = g^{(i)}_{ba} & i = 1, 2, 3, 5 \\
g^{(5)}_{abcd} = g^{(5)}_{bacd} = g^{(5)}_{abdc} & g^{(6)}_{abcd} = g^{(6)}_{bacd} &
\end{array}
\tag{3.2.20}
$$

and one can suppose that $g^{(4)}_{abcd}$ (resp. $g^{(7)}_{abcd}$) are completely antisymmetric (resp. symmetric) in all indices.

(ii) By integration over x some of the linear independence is lost in the adiabatic limit. (In the language of axiom (3.1.11), some of the terms can be grouped in total divergences.) Namely, all the conclusions from [16] stay true and we have in the end:

- one can eliminate $T^{(3)}$ by redefining the constants $f^{(2)}_{abc}$ and $f^{(4)}_{abc}$;
- one can eliminate $T^{(5)}$ by redefining the constants $f^{(1)}_{abc}$;
- one can eliminate $T^{(12)}$ and $T^{(13)}$ by redefining the constants $g^{(2)}_{ab}$;
- one can eliminate $T^{(10)}$ and $T^{(15)}$ using the equation of motion (2.1.20) and (2.2.11);
- $T^{(14)}$ is null in the adiabatic limit;
- one can eliminate $T^{(5)''}$ by redefining the constants $h^{(4)}_{ab}$;
- one can choose the constants $f^{(7)}_{abc}$ such that they verify

$$f^{(7)}_{abc} = -f^{(7)}_{bac} \tag{3.2.21}$$

if one modifies $f^{(6)}_{abc}$ appropriately;

- one can eliminate $T^{(21)}$ by redefining the constants $h^{(3)}_{ab}$.

(iii) Some of the remaining expressions are of the form $d_Q O$ so they do not count. Namely:

- we have

$$d_Q : (\partial_\mu A^\mu_a + m_a \Phi_a)\tilde{u}_b :=: (\partial_\mu A^\mu_a + m_a \Phi_a)(\partial_\mu A^\mu_b + m_b \Phi_b) :$$

so we can give up the expressions $T^{(11)}$ if we modify appropriately the expressions $h^{(3)}_{ab}$ and $f^{(6)}_{abc}$ conveniently; afterwards one can trade off $f^{(6)}_{abc}$ modifying $f^{(7)}_{abc}$ by integration by parts as explained above;

- we have

$$d_Q : \tilde{u}_a \Phi_b := -\mathrm{i} : (\partial_\mu A^\mu_a + m_a \Phi_a)\Phi_b : +\mathrm{i} m_b : \tilde{u}_a u_b :$$

so we can eliminate the expression $T^{(4)''}$ if we redefine the expressions $h^{(3)}_{ab}$ and $h^{(2)}_{ab}$;

- if the constants g_{abc} are chosen antisymmetric in the indices a and c, then we have

$$d_Q g_{abc} : \tilde{u}_a u_b \tilde{u}_c := 2\mathrm{i} g_{abc} : (\partial_\mu A^\mu_a + m_a \Phi_a) u_b \tilde{u}_c :$$

so, it follows that if we modify conveniently the constants $h^{(2)}_{abc}$ we can impose

$$f^{(4)}_{abc} = f^{(4)}_{cba}; \tag{3.2.22}$$

- we have

$$d_Q : \Phi_a \Phi_b \tilde{u}_c := \mathrm{i} m_a : u_a \Phi_b \tilde{u}_c : +\mathrm{i} m_b : \Phi_a u_b \tilde{u}_c : - : \Phi_a \Phi_b (\partial_\mu A^\mu_c + m_c \Phi_c) :$$

so we can give up the term $T^{(16)}$ if we modify conveniently the constants $h^{(2)}_{abc}$ and $h^{(3)}_{abc}$.

(iv) As a conclusion, we can keep in T_1 only the expressions $T^{(1)'} - T^{(3)'}$, $T^{(1)''} - T^{(3)''}$, $T^{(1)}$, $T^{(2)}$, $T^{(4)}$, $T^{(6)} - T^{(9)}$ and $T^{(17)} - T^{(20)}$ with the appropriate symmetry properties.

We compute now the expression $d_Q T_1$. The expression (4.2.7) from [16] receives new contributions:

$$\begin{aligned}
d_Q T_1 = d_Q T^{(0)}_1 &- \mathrm{i} h^{(2)}_{ab} m_a : \Phi_a u_b : + \mathrm{i} f^{(2)}_{abc} m_c : A^\mu_a u_b \partial_\mu \Phi_c : - \mathrm{i} f^{(4)}_{abc} m_c : \partial_\mu A^\mu_b u_b \Phi_c : \\
&- \mathrm{i} g^{(2)}_{abcd} m_d : A_{a\mu} A^\mu_b u_c \Phi_d : - 2\mathrm{i} g^{(3)}_{abcd} m_d : u_a u_b \tilde{u}_c \Phi_d : \\
&+ \mathrm{i} f^{(1)}_{abc} m^2_c : A_{a\mu} u_b A^\mu_c : \\
&- \mathrm{i} f^{(4)}_{abc} m^2_a : u_a u_b \tilde{u}_c : + 2\mathrm{i} h^{(3)}_{ab} m_a : u_a \Phi_b : \\
&+ \mathrm{i} f^{(7)}_{abc} [m_a : u_a \partial_\mu \Phi_b A^\mu_c : + m_b : \Phi_a \partial_\mu u_b A^\mu_c : + : \Phi_a \partial^\mu \Phi_b \partial_\mu u_c :] \\
&+ \mathrm{i} h^{(1)}_{abc} [m_a : u_a A_{b\mu} A^\mu_c : + 2 : \Phi_a A^\mu_b \partial_\mu u_c :] \\
&+ \mathrm{i} h^{(2)}_{abc} [m_a : u_a \tilde{u}_b u_c : - : \Phi_a (\partial_\mu A^\mu_b + m_b \Phi_b) u_c :] \\
&+ 3\mathrm{i} h^{(3)}_{abc} m_a : u_a \Phi_b \Phi_c : + 2\mathrm{i} g^{(5)}_{abcd} [m_a : u_a \Phi_b A_{c\mu} A^\mu_d : + : \Phi_a \Phi_b A^\mu_c \partial_\mu u_d :] \\
&+ \mathrm{i} g^{(6)}_{abcd} [2 m_a : u_a \Phi_b \tilde{u}_c u_d : - : \Phi_a \Phi_b (\partial_\mu A^\mu_c + m_c \Phi_c) u_d :] \\
&+ 4\mathrm{i} g^{(7)}_{abcd} m_a : u_a \Phi_b \Phi_c \Phi_d : .
\end{aligned} \tag{3.2.23}$$

Here

$$\begin{aligned}
d_Q T^{(0)}_1 = \mathrm{i}\partial_\mu [& 2 h^{(1)}_{ab} : A^\mu_a u_b : + (f^{(1)}_{abc} - f^{(1)}_{cba}) : u_a A_{b\nu} \partial^\nu A^\mu_c : + f^{(1)}_{bac} : u_a A_{b\nu} \partial^\mu A^\nu_c : \\
&+ f^{(1)}_{bca} : \partial_\nu u_a A^\nu_b A^\mu_c : - f^{(1)}_{cba} : u_a \partial^\nu A_{b\nu} A_{c\nu} :] \\
&- \mathrm{i}(2 h^{(1)}_{ab} + h^{(2)}_{ab}) : \partial_\mu A^\mu_a u_b : + \mathrm{i}(f^{(1)}_{cba} - f^{(1)}_{abc}) : u_a \partial_\mu A_{b\nu} \partial^\nu A^\mu_c : \\
&+ \mathrm{i}(f^{(1)}_{abc} - f^{(1)}_{bac} - f^{(1)}_{cba} + f^{(2)}_{cba}) : \partial_\mu \partial_\nu A^\mu_a A^\nu_b u_c : \\
&+ \mathrm{i}(f^{(1)}_{abc} + f^{(4)}_{acb}) : \partial_\mu A^\mu_a \partial_\nu A^\nu_b u_c :
\end{aligned}$$

$$-\mathrm{i}f^{(1)}_{acb} : \partial_\nu A_{a\mu}\partial^\nu A^\mu_b u_c : +\mathrm{i}f^{(2)}_{abc} : \partial^\mu u_a u_b \partial_\mu \tilde{u}_c :$$
$$+4\mathrm{i}g^{(1)}_{abcd} : \partial_\mu u_a A^\mu_b A_{c\nu} A^\nu_d :$$
$$+\mathrm{i}g^{(2)}_{abcd}(2 : \partial_\mu u_a A^\mu_b u_c \tilde{u}_d : + : A_{a\mu} A^\mu_b u_c \partial_\rho A^\rho_d :)$$
$$-2\mathrm{i}g^{(3)}_{abcd} : u_a u_b \partial_\mu A^\mu_c \tilde{u}_d : -4\mathrm{i}g^{(4)}_{abcd}\varepsilon_{\mu\nu\rho\sigma} : \partial^\mu u_a A^\nu_b A^\rho_c A^\sigma_d : \qquad (3.2.24)$$

is the expression (4.2.7) from [16], i.e. the expression $\mathrm{d}_Q T_1$ for zero-mass bosons and without scalar ghosts, the next terms having various origins: the modification of the BRST transformation (3.1.9), the modification of the equation of motion (3.1.5) and the new terms $T^{(3)'}$, $T^{(1)''}$–$T^{(3)''}$ and $T^{(17)}$–$T^{(20)}$ considered in the expression of T_1. We impose the condition of factorization to the physical space (3.1.12) for the case $n = 1$:

$$\int_{\mathbb{R}^4} \mathrm{d}x\, g_\epsilon(x)\, \mathrm{d}_Q T_1\,(x)\bigg|_{\mathrm{Ker}(Q)} = \mathrm{O}(\epsilon). \qquad (3.2.25)$$

It is not very hard to see that all the conclusions from [16] remain true, i.e. the constants $f_{abc} \equiv f^{(1)}_{abc}$ are completely antisymmetric:

$$f^{(2)}_{abc} = -f_{abc} \qquad f^{(4)}_{abc} = 0$$
$$g^{(i)}_{abcd} = 0 \quad i = 1,2,3,4 \qquad 2h^{(1)}_{ab} + h^{(2)}_{ab} = 0. \qquad (3.2.26)$$

Moreover, we get

$$2h^{(1)}_{abc} m_a = f_{bac}(m_b^2 - m_c^2) \qquad \forall a,b,c = 1,\ldots,r \qquad (3.2.27)$$
$$h_{ab} m_a = 2h^{(3)}_{ab} m_b \qquad \forall a,b = 1,\ldots,r \qquad (3.2.28)$$
$$g^{(i)}_{abcd} = 0 \qquad i = 5,6 \qquad (3.2.29)$$

and the expressions $g^{(7)}_{abcd}$ can be non-zero *iff* $m_a = m_b = m_c = m_d = 0$.

It remains to perform some integrations by parts into the remaining expression and to obtain

$$\mathrm{d}_Q T_1 = \mathrm{i}\partial_\mu[\cdots + (2h^{(1)}_{cab} + f^{(7)}_{cba} m_b) : A^\mu_a u_b \Phi_c : + f^{(7)}_{abc} : \Phi_a \partial^\mu \Phi_b u_c :]$$
$$-\mathrm{i}(f_{abc} m_c^2 + h^{(2)}_{acb} m_a) : u_a u_b \tilde{u}_c :$$
$$+\mathrm{i}(-f_{abc} m_c + 2f^{(7)}_{bca} m_b - 2h^{(1)}_{cab}) : A^\mu_a u b \partial_\mu \Phi_c :$$
$$-\mathrm{i}(2h^{(1)}_{acb} + f^{(7)}_{abc} m_b + h^{(2)}_{acb}) : \Phi_a u_b \partial_\mu A^\mu_c :$$
$$-\mathrm{i}[h^{(2)}_{abc} m_b - 3h^{(3)}_{abc} m_c - f^{(7)}_{abc}(m^H_b)^2] : \Phi_a \Phi_b u_c : \qquad (3.2.30)$$

where by $\cdots$ we mean the expression obtained if all the masses are zero and there are no scalar ghosts (see (3.2.24) above). The divergence gives a contribution of order ε in (3.2.25) and the other terms can be computed on vectors from $\mathcal{H}'$. In this way we see that we get independent conditions from each term in the preceding formula, i.e.

$$2f_{abc} m_c^2 = h^{(2)}_{bca} m_b - h^{(2)}_{acb} m_a \qquad \forall a,b,c = 1,\ldots,r \qquad (3.2.31)$$
$$2h^{(1)}_{cab} = -f_{abc} m_c + 2f^{(7)}_{bca} m_b \qquad \forall a,b,c = 1,\ldots,r \qquad (3.2.32)$$
$$h^{(2)}_{abc} = -2h^{(1)}_{abc} - f^{(7)}_{acb} m_b \qquad \forall a,b,c = 1,\ldots,r \qquad (3.2.33)$$

and

$$6h^{(3)}_{abc} m_c = f^{(7)}_{abc}[(m^H_a)^2 - (m^H_b)^2] + h^{(2)}_{bac} m_a + h^{(2)}_{abc} m_b \qquad \forall a,b,c = 1,\ldots,r. \qquad (3.2.34)$$

We exploit completely the system of equations (3.2.27), (3.2.31)–(3.2.34). It is obvious that in order to obtain the statement of the theorem we should redefine $f^{(7)}_{abc} \rightarrow f'_{abc}$,

$h^{(3)}_{abc} \rightarrow f''_{abc}$ and $h^{(3)}_{ab} \rightarrow h'_{ab}$. If we take the symmetric (resp. antisymmetric) part in a and b of relation (3.2.32) we get an explicit expression for $h^{(1)}_{cab}$:

$$h^{(1)}_{cab} = \frac{1}{2}(f'_{bca}m_b + f'_{acb}m_a) \qquad \forall a, b, c = 1, \ldots, r \tag{3.2.35}$$

and respectively the consistency relation (3.2.12). One substitutes this result into equations (3.2.27) (resp. (3.2.33)) and gets an identity (resp. an explicit expression for $h^{(2)}_{abc}$)

$$h^{(2)}_{abc} = f'_{abc}m_b \qquad \forall a, b, c = 1, \ldots, r. \tag{3.2.36}$$

Next, from (3.2.34) for $m_c = 0$ we get the consistency relation (3.2.11) and for $m_c \neq 0$ we obtain the expression (3.2.13).

Finally, from (3.2.28) we immediately get the consistency relation (3.2.15) and the explicit expression (3.2.16). If the expressions for $h^{(1)}_{abc}$ and $h^{(2)}_{abc}$ are substituted into the generic expression for T_1 we get the formula from the statement.

(vi) It remains to prove that the expression from the statement cannot be of the type $d_Q O$ and this can be easily done. □

Remark 3.2. It is a remarkable fact that we get in a natural way mass relations of the type (3.2.9). This relation is non-trivial iff there are simultaneously massive and massless bosons in the model. In this case, we can reformulate this relation as follows: if $f_{abc} \neq 0$ and $m_c = 0$ then necessarily we have $m_a = m_b$. In particular, this is the cause of the equality of the masses of the two heavy W bosons in the standard model.

The relation (3.2.12) can be completely exploited.

Corollary 3.3. *The following relations are true:*

$$f'_{abc} = \frac{m_a^2 + m_b^2 - m_c^2}{2m_am_b} f_{abc} \qquad \forall a, b, c = 1, \ldots, r \qquad \text{s.t.} \quad m_a \neq 0 \qquad m_b \neq 0 \tag{3.2.37}$$

and

$$f'_{abc} = m_c g_{abc} \qquad \forall a, b, c = 1, \ldots, r \qquad \text{s.t.} \quad m_a = 0 \qquad m_b \neq 0. \tag{3.2.38}$$

Here the constants g_{abc} are constrained only by the symmetry property in the last two indices

$$g_{abc} = g_{acb}. \tag{3.2.39}$$

These two relations are completely equivalent to the relation (3.2.12) so, in particular, the constants f'_{abc} remain arbitrary for $m_a = m_b = 0$.

Proof. The first relation can be obtained if we multiply (3.2.12) by m_c and perform two cyclic permutations. Combining the three relations in a convenient way one gets (3.2.37). The relation (3.2.38) follows from (3.2.12) if we consider the case $m_c = 0$. □

Corollary 3.4. *In the condition of the preceding theorem, one has*

$$d_Q T_1(x) = i\partial_\mu T_1^\mu(x) \tag{3.2.40}$$

where

$$T_1^\mu \equiv T_{11}^\mu + T_{12}^\mu + T_{13}^\mu \tag{3.2.41}$$

and the expression from this formula is defined as follows:

$$T_{11}^{\mu} \equiv f_{abc}(: u_a A_{b\nu} F_c^{\nu\mu} : -\tfrac{1}{2} : u_a u_b \partial^\mu \tilde{u}_c :) \tag{3.2.42}$$

$$T_{12}^{\mu} \equiv f'_{abc}(m_a : A_a^\mu \Phi_b u_c : + : \Phi_a \partial^\mu \Phi_b u_c :) \tag{3.2.43}$$

and

$$T_{13}^{\mu} \equiv 2h_{ab} : A_a^\mu u_b : . \tag{3.2.44}$$

Moreover, we have the followoing proposition.

Proposition 3.5. *The expression T_1 from the preceding theorem verifies the unitarity condition*

$$T_1(x)^\dagger = T_1(x)$$

if and only if the constants f_{abc}, f'_{abc}, f''_{abc}, h_{ab} and h'_{ab} have real values.

The proof is very simple and relies on relations (3.1.8). To study the causality axiom in the first order of the perturbation theory, one has to investigate some causal distributions and some relations between them. We have the following proposition.

Proposition 3.6. *The following distributions are well defined and have causal support:*

$$\begin{aligned}
&D_{m_a m_b}(x) \equiv D_{m_a}^{(+)}(x) D_{m_b}^{(+)}(x) - (+ \to -)\\
&D_{m_a m_b;\mu\nu} \equiv \left[D_{m_a}^{(+)}(x) \frac{\partial^2}{\partial x^\mu \partial x^\nu} D_{m_b}^{(+)}(x) \right.\\
&\qquad \left. - \frac{\partial}{\partial x^\mu} D_{m_a}^{(+)}(x) \frac{\partial}{\partial x^\nu} D_{m_b}^{(+)}(x) + (a \leftrightarrow b) \right] - (+ \to -)\\
&D_{m_a m_b;\mu} \equiv \frac{\partial}{\partial x^\mu} D_{m_a}^{(+)}(x) D_{m_b}^{(+)}(x) + (+ \to -)\\
&D_{m_a m_b m_c}(x) \equiv D_{m_a}^{(+)}(x) D_{m_b}^{(+)}(x) D_{m_c}^{(+)}(x) + (+ \to -)\\
&D_{m_a m_b;m_c}(x) \equiv [\partial_\mu D_{m_a}^{(+)}(x) \partial^\mu D_{m_b}^{(+)}(x)] D_{m_c}^{(+)}(x) + (+ \to -)\\
&D_{m_a m_b m_c m_d}(x) \equiv D_{m_a}^{(+)}(x) D_{m_b}^{(+)}(x) D_{m_c}^{(+)}(x) D_{m_d}^{(+)}(x) - (+ \to -).
\end{aligned} \tag{3.2.45}$$

Moreover, they verify the following relations:

$$\begin{aligned}
&\frac{\partial}{\partial x_\nu} D_{m_a m_b;\mu\nu} = (m_b^2 - m_a^2)[D_{m_a m_b;\mu} - (a \leftrightarrow b)]\\
&\frac{\partial}{\partial x_\mu} D_{m_a m_b;\mu} = \tfrac{1}{2}(\Box + m_b^2 - m_a^2) D_{m_a m_b}.
\end{aligned} \tag{3.2.46}$$

Finally we have the following.

Proposition 3.7. *The expression T_1 determined in the preceding theorem verifies the causality condition*

$$[T_1(x), T_1(y)] = 0 \qquad \forall x, y \in \mathbb{R}^4 \qquad \text{s.t.} \quad (x-y)^2 < 0.$$

One must determine the commutator appearing in the left-hand side. The computations are similar to the one from [16] and we do not give them here. We only mention that the commutator involves the distributions listed in (3.2.45), which have causal support.

We can now give a generic form for the distribution T_2. We split causally the commutator $[T_1(x), T_1(y)]$ according to the prescription of Epstein and Glaser and include the most general finite arbitrariness of the decomposition taking into account general considerations explained in [16]. First we note that we have the following.

Proposition 3.8. *The distributions listed in (3.2.45) admit causal splittings which preserve Lorentz covariance. Moreover, the splitting can be chosen such that it will preserve the properties (3.2.46), i.e. we can arrange it such that we have*

$$\begin{aligned} \frac{\partial}{\partial x_\nu} D^{\mathrm{ret(adv)}}_{m_a m_b;\mu\nu} &= (m_b^2 - m_a^2)[D^{\mathrm{ret(adv)}}_{m_a m_b;\mu} - (a \leftrightarrow b)] \\ \frac{\partial}{\partial x_\mu} D^{\mathrm{ret(adv)}}_{m_a m_b;\mu} &= \frac{1}{2}(\Box + m_b^2 - m_a^2) D^{\mathrm{ret(adv)}}_{m_a m_b}. \end{aligned} \tag{3.2.47}$$

So we can provide now the generic expression of the distribution T_2. The expression is extremely long, but we provide it because it provides the easiest way to compute explicit effects in a concrete theory, like the standard model. (For this, one should include, of course, the lepton fields.) We will denote by $D^F_m(x)$, $D^F_{m_a m_b}(x)$, $D^F_{m_a m_b;\mu\nu}$, $D^F_{m_a m_b;\mu}$, $D^F_{m_a m_b m_c}(x)$ and $D^F_{m_a m_b;m_c}(x)$ the corresponding Feynman propagators and observe that they verify equations of the same type as those from the preceding proposition. We have the following proposition, obtained by long and tedious computations.

Proposition 3.9. *The generic form of the distribution T_2 is a sum between the non-contracted term* $: T_1(x)T_1(y) :$, *the expressions*

$$\begin{aligned} T_2^0(x,y) = {} & -f_{cab} f_{cde} D^F_{m_c}(x-y)\{: A_{a\nu}(x) F_b^{\nu\mu}(x) A_d^\rho(y) F_{e\rho\mu}(y) : \\ & + : u_a(x)\partial_\mu \tilde{u}_b(x) u_d(y) \partial^\mu \tilde{u}_e(y) : - [A_{a\nu}(x) F_b^{\nu\mu}(x) u_d(y) \partial_\mu \tilde{u}_e(y) : + (x \leftrightarrow y)]\} \\ & + \tfrac{1}{2} f_{abc} f_{dbc} D^F_{m_b m_c}(x-y) : F_a^{\nu\mu}(x) F_{d\nu\mu}(y) : \\ & + f_{cab} f_{cde} \frac{\partial}{\partial x^\rho} D^F_{m_c}(x-y)\{[: A_{a\nu}(x) F_b^{\nu\mu}(x) A_{d\mu}(y) A_e^\rho(y) : \\ & + : A_a^\mu(x) A_b^\rho(x) u_d(y) \partial_\mu \tilde{u}_e(y) : \\ & + : A_a^\mu(x) \partial_\mu \tilde{u}_b(x) A_d^\rho(y) u_e(y) :] - (x \leftrightarrow y)\} \\ & + f_{cab} f_{cde} \frac{\partial^2}{\partial x^\mu \partial x^\nu} D^F_{m_c}(x-y) : A_{a\rho}(x) A_b^\mu(x) A_d^\rho(y) A_e^\nu(y) : \\ & + f_{abc} f_{dbc} [m_b^2 g_{\mu\nu} D^F_{m_b m_c}(x-y) - 2 D^F_{m_b m_c;\mu\nu}(x-y)] : A_a^\mu(x) A_d^\nu(y) : \\ & + f_{abc} f_{dbc} D^F_{m_b,m_c;\rho}(x-y)\{[: A_{a\mu}(x) F_d^{\mu\rho}(y) : + : u_a(x) \partial^\rho \tilde{u}_d(y) :] - (x \leftrightarrow y)\} \\ & - f_{abc} f_{abc} (\tfrac{1}{3}\Box - 2m_a^2) D^F_{m_a m_b m_c}(x-y) \mathbf{1} \\ & + f_{cab} f'_{dec} D^F_{m_c}(x-y)\{[: A_{a\mu}(x) F_b^{\mu\nu}(x) \Phi_d(y) \partial_\nu \Phi_e(y) : \\ & + : u_a(x) \partial_\mu \tilde{u}_b(x) \Phi_d(y) \partial^\mu \Phi_e(y) :] + (x \leftrightarrow y)\} \\ & + f_{cab} f'_{dec} \frac{\partial}{\partial x^\rho} D^F_{m_c}(x-y)[: A_a^\mu(x) A_b^\rho(x) \Phi_d(y) \partial_\mu \Phi_e(y) : - (x \leftrightarrow y)] \\ & - 2 f_{abc} h^{(1)}_{dec} D^F_{m_c}(x-y)\{[: A_{a\mu}(x) F_b^{\mu\nu}(x) \Phi_d(y) A_{e\nu}(y) : \\ & - : u_a(x) \partial_\mu \tilde{u}_b(x) \Phi_d(y) A_e^\mu(y) :] + (x \leftrightarrow y)\} \\ & - 2 f_{abc} h^{(1)}_{dec} \frac{\partial}{\partial x^\rho} D^F_{m_c}(x-y)[: A_{a\mu}(x) A_b^\rho(x) \Phi_d(y) A_e^\mu(y) : - (x \leftrightarrow y)] \\ & - f_{cab} h^{(2)}_{dce} D^F_{m_c}(x-y)[: A_a^\mu(x) \partial_\mu \tilde{u}_b(x) \Phi_d(y) u_e(y) : + (x \leftrightarrow y)] \\ & - f_{cab} h^{(2)}_{dec} \frac{\partial}{\partial x^\rho} D^F_{m_c}(x-y)[: A_a^\rho(x) u_b(x) \Phi_d(y) \tilde{u}_e(y) : - (x \leftrightarrow y)] \\ & - f_{cab} h^{(2)}_{dcb} D^F_{m_b m_c;\mu}(x-y)[: A_a^\mu(x) \Phi_d(y) : - (x \leftrightarrow y)] \\ & + f'_{cab} f'_{cde} D^F_{m_c^*}(x-y) : \partial_\mu \Phi_a(x) A_b^\mu(x) \partial_\nu \Phi_d(y) A_e^\nu(y) : \end{aligned}$$

$$+f'_{cab}f'_{cde}\left[\frac{\partial}{\partial x^{\rho}}D^F_{m^*_c}(x-y):\partial_\mu\Phi_a(x)A^\mu_b(x)\Phi_d(y)A^\rho_e(y):\right.$$
$$\left.-\frac{\partial}{\partial x^{\mu}}D^F_{m^*_c}(x-y):\Phi_a(x)A^\mu_b(x)\partial_\rho\Phi_d(y)A^\rho_e(y):\right]$$
$$-f'_{cab}f'_{cde}\frac{\partial}{\partial x^\mu\partial x^\nu}D^F_{m^*_c}(x-y):\Phi_a(x)A^\mu_b(x)\Phi_d(y)A^\nu_e(y):$$
$$-f'_{abc}f'_{dec}D^F_{m_c}(x-y):\Phi_a(x)\partial_\mu\Phi_b(x)\Phi_d(y)\partial^\mu\Phi_e(y):$$
$$-f'_{abc}f'_{abd}D^F_{m^*_am^*_b;\mu\nu}(x-y):A^\mu_c(x)A^\nu_d(x):$$
$$-f'_{abc}f'_{adc}D^F_{m^*_am_c}(x-y):\partial_\mu\Phi_b(x)\partial^\mu\Phi_d(x):$$
$$-f'_{abc}f'_{adc}D^F_{m_cm^*_a;\mu}(x-y)[:\partial^\mu\Phi_b(x)\Phi_d(x):-(x\leftrightarrow y)]$$
$$-f'_{abc}f'_{adc}(m^*_a)^2D^F_{m^*_am_c}(x-y):\Phi_b(x)\Phi_d(x):$$
$$-f'_{abc}f'_{abc}[D^F_{m^*_am^*_b;m_c}(x-y)+(m^*_a)^2D^F_{m^*_am^*_bm_c}(x-y)]\mathbf{1}$$
$$-2f'_{abc}h^{(1)}_{dec}D^F_{m_c}(x-y)[:\Phi_a(x)\partial_\mu\Phi_b(x)\Phi_d(y)A^\mu_e(y):+(x\leftrightarrow y)]$$
$$+f'_{cab}h^{(1)}_{cde}D^F_{m^*_c}(x-y)[:\partial_\mu\Phi_a(x)A^\mu_b(x)A_{d\rho}(y)A^\rho_e(y):+(x\leftrightarrow y)]$$
$$-f'_{cab}h^{(1)}_{cde}\frac{\partial}{\partial x^\mu}D^F_{m^*_c}(x-y)[:\Phi_a(x)A^\mu_b(x)A_{d\rho}(y)A^\rho_e(y):-(x\leftrightarrow y)]$$
$$+2f'_{abc}h^{(1)}_{bcd}D^F_{m^*_bm_c}(x-y)[:\partial_\mu\Phi_a(x)A^\mu_d(y):+(x\leftrightarrow y)]$$
$$-2f'_{abc}h^{(1)}_{bcd}D^F_{m_cm^*_b;\mu}(x-y)[:\Phi_a(x)A^\mu_d(y):-(x\leftrightarrow y)]$$
$$+f'_{cab}h^{(2)}_{cde}D^F_{m^*_c}(x-y)[:\partial_\mu\Phi_a(x)A^\mu_b(x)\tilde{u}_d(y)u_e(y):+(x\leftrightarrow y)]$$
$$-f'_{cab}h^{(2)}_{cde}\frac{\partial}{\partial x^\rho}D^F_{m^*_c}(x-y)[:\Phi_a(x)A^\rho_b(x)\tilde{u}_d(y)u_e(y):-(x\leftrightarrow y)]$$
$$+3f'_{cab}f''_{cde}D^F_{m^*_c}(x-y)[:\partial_\mu\Phi_a(x)A^\mu_b(x)\Phi_d(y)\Phi_e(y):+(x\leftrightarrow y)]$$
$$-3f'_{cab}f''_{cde}\frac{\partial}{\partial x^\rho}D^F_{m^*_c}(x-y)[:\Phi_a(x)A^\rho_b(x)\Phi_d(y)\Phi_e(y):-(x\leftrightarrow y)]$$
$$+6f'_{bca}f''_{bcd}D^F_{m^*_cm^*_b;\rho}(x-y)[:A^\rho_a(x)\Phi_d(y):-(x\leftrightarrow y)]$$
$$+4f'_{cab}g_{cdef}D^F_{m^*_c}(x-y)[:\partial_\mu\Phi_a(x)A^\mu_b(x)\Phi_d(y)\Phi_e(y)\Phi_f(y):+(x\leftrightarrow y)]$$
$$-4f'_{cab}g_{cdef}\frac{\partial}{\partial x^\rho}D^F_{m^*_c}(x-y)[:\Phi_a(x)A^\rho_b(x)\Phi_d(y)\Phi_e(y)\Phi_f(y):-(x\leftrightarrow y)]$$
$$-12f'_{bca}g_{bcde}D^F_{m^*_bm^*_c;\rho}(x-y)[:A^\rho_a(x)\Phi_d(y)\Phi_e(y):-(x\leftrightarrow y)]$$
$$+h^{(1)}_{cab}h^{(1)}_{cde}D^F_{m^*_c}(x-y):A_{a\mu}(x)A^\mu_b(x)A_{d\nu}(y)A^\nu_e(y):$$
$$-4h^{(1)}_{cab}h^{(1)}_{dbe}D^F_{m_b}(x-y):\Phi_a(x)A^\mu_c(x)\Phi_d(y)A_{e\mu}(y):$$
$$-4h^{(1)}_{abc}h^{(1)}_{abd}D^F_{m^*_am_b}(x-y):A^\mu_c(x)A_{d\mu}(y):+2h^{(1)}_{abc}h^{(1)}_{dbc}D^F_{m_bm_c}(x-y)$$
$$:\Phi_a(x)\Phi_d(y)+2h^{(1)}_{abc}h^{(1)}_{abc}D^F_{m^*_am_bm_c}(x-y)\mathbf{1}$$
$$+h^{(1)}_{cab}h^{(2)}_{cde}D^F_{m^*_c}(x-y)[:A_{a\mu}(x)A^\mu_b(x)\tilde{u}_d(y)u_e(y):+(x\leftrightarrow y)]$$
$$+3h^{(1)}_{cab}h^{(3)}_{cde}D^F_{m^*_c}(x-y)[:A_{a\mu}(x)A^\mu_b(x)\Phi_d(y)\Phi_e(y):+(x\leftrightarrow y)]$$
$$+4h^{(1)}_{cab}g_{cdef}D^F_{m^*_c}(x-y)[:A_{a\mu}(x)A^\mu_b(x)\Phi_d(y)\Phi_e(y)\Phi_f(y):+(x\leftrightarrow y)]$$
$$+h^{(2)}_{abc}h^{(2)}_{ade}D^F_{m^*_a}(x-y):\tilde{u}_b(x)u_c(x)\tilde{u}_d(y)u_e(y):$$

$$+h^{(2)}_{abc}h^{(2)}_{deb}D^F_{m_b}(x-y)[:\Phi_a(x)u_c(x)\Phi_d(y)\tilde{u}_e(y):-(x\leftrightarrow y)]$$
$$+h^{(2)}_{abc}h^{(2)}_{aeb}D^F_{m^*_a m_b}(x-y)[:\tilde{u}_e(x)u_c(y):+(x\leftrightarrow y)]$$
$$-h^{(2)}_{abc}h^{(2)}_{dbc}D^F_{m_b m_c}(x-y):\Phi_a(x)\Phi_d(y):-h^{(2)}_{abc}h^{(2)}_{acb}D^F_{m^*_a m_b m_c}(x-y)\mathbf{1}$$
$$+3h^{(2)}_{cab}f^{''}_{cde}D^F_{m^*_c}(x-y)[:\tilde{u}_a(x)u_b(x)\Phi_d(y)\Phi_e(y):+(x\leftrightarrow y)]$$
$$+4h^{(2)}_{cab}g_{cdef}D^F_{m^*_c}(x-y)[:\tilde{u}_a(x)u_b(x)\Phi_d(y)\Phi_e(y)\Phi_f(y):+(x\leftrightarrow y)]$$
$$+9f''_{cab}f^{''}_{cde}D^F_{m^*_c}(x-y):\Phi_a(x)\Phi_b(x)\Phi_d(y)\Phi_e(y):$$
$$+18f^{''}_{abc}f^{''}_{abd}D^F_{m^*_a m^*_b}(x-y):\Phi_c(x)\Phi_d(y):+18f^{''}_{abc}f^{''}_{abc}D^F_{m^*_a m^*_b m^*_c}(x-y)\mathbf{1}$$
$$+12f''_{cab}g_{cdef}D^F_{m^*_c}(x-y)[:\Phi_a(x)\Phi_b(x)\Phi_d(y)\Phi_e(y)\Phi_f(y):+(x\leftrightarrow y)]$$
$$+18f^{''}_{abc}g_{bcde}D^F_{m^*_b m^*_c}(x-y)[:\Phi_a(x)\Phi_d(y)\Phi_e(y):+(x\leftrightarrow y)]$$
$$+24f^{''}_{bcd}g_{abcd}D^F_{m^*_b m^*_c m^*_d}(x-y)[\Phi_a(x)+(x\leftrightarrow y)]$$
$$+16g_{abcd}g_{defh}D^F_{m^*_d}(x-y):\Phi_a(x)\Phi_b(x)\Phi_c(x)\Phi_e(y)\Phi_f(y)\Phi_h(y):$$
$$+144g_{abcd}g_{cdef}D^F_{m^*_c m^*_d}(x-y):\Phi_a(x)\Phi_b(x)\Phi_e(y)\Phi_f(y):$$
$$+576g_{abcd}g_{bcde}D^F_{m^*_b m^*_c m^*_d}(x-y):\Phi_a(x)\Phi_e(y):$$
$$+24g_{abcd}g_{abcd}D^F_{m^*_a m^*_b m^*_c m^*_d}(x-y)\mathbf{1}. \qquad (3.2.48)$$

$$T^h_2(x,y)=-2f_{cab}h_{cd}D^F_{m_c}(x-y)\{[:A_{a\nu}(x)F^{\nu\mu}_b(x)A_{d\mu}(y):-:u_a(x)\partial_\mu\tilde{u}_b(x)A^\mu_d(y):$$
$$-:A^\mu_a(x)\partial_\mu\tilde{u}_b(y)u_d(y):]+(x\leftrightarrow y)]\}$$
$$-2f_{cab}h_{cd}\frac{\partial}{\partial x^\rho}D^F_{m_c}(x-y)\{[:A_{a\mu}(x)A^\rho_b(x)A^\mu_d(y):-$$
$$:A^\rho_a(x)u_b(x)\tilde{u}_d(y):]-(x\leftrightarrow y)]\}$$
$$-2f'_{abc}h_{cd}D^F_{m_c}(x-y)[:\Phi_a(x)\partial_\mu\Phi_b(x)A^\mu_d(y):+(x\leftrightarrow y)]$$
$$+2f'_{abc}h'_{cd}D^F_{m^*_c}(x-y)[:\partial_\mu\Phi_a(x)A^\mu_b(x)\Phi_d(y):+(x\leftrightarrow y)]$$
$$-2f'_{cab}h'_{cd}\frac{\partial}{\partial x^\rho}D^F_{m^*_c}(x-y)[:\Phi_a(x)A^\rho_c(x)\Phi_d(y):-(x\leftrightarrow y)]$$
$$+2f'_{bca}h'_{bc}D^F_{m^*_c m^*_b;\rho}(x-y)[A^\rho_a(x)-(x\leftrightarrow y)]$$
$$-4h^{(1)}_{cab}h_{cd}D^F_{m_c}(x-y)[:\Phi_a(x)A^\mu_b(x)A_{d\mu}(y):+(x\leftrightarrow y)]$$
$$+16h^{(1)}_{cab}h_{cd}D^F_{m_c}(x-y)\Phi_a(x)$$
$$+2h^{(1)}_{cab}h'_{cd}D^F_{m^*_c}(x-y)[:A_{a\mu}(x)A^\mu_b(x)\Phi_d(y):+(x\leftrightarrow y)]$$
$$+2h^{(2)}_{acb}h_{cd}D^F_{m_c}(x-y)[:\Phi_a(x)u_b(x)\tilde{u}_d(y):+(x\leftrightarrow y)]$$
$$-2h^{(2)}_{abc}h_{cd}D^F_{m_c}(x-y)[:\Phi_a(x)\tilde{u}_b(x)u_d(y):+(x\leftrightarrow y)]$$
$$+4h^{(2)}_{abc}h_{bc}D^F_{m_b m_c}(x-y)\Phi_a(x)$$
$$+2h^{(2)}_{cab}h'_{cd}D^F_{m^*_c}(x-y)[:\tilde{u}_a(x)u_b(x)\Phi_d(y):+(x\leftrightarrow y)]$$
$$+6h^{(3)}_{cab}h'_{cd}D^F_{m^*_c}(x-y)[:\Phi_a(x)\Phi_b(x)\Phi_d(y):+(x\leftrightarrow y)]$$
$$+6h^{(3)}_{abc}h'_{bc}D^F_{m^*_b m^*_c}(x-y)\Phi_a(x)$$
$$+8g_{abcd}h'_{de}D^F_{m^*_d}(x-y)[:\Phi_a(x)\Phi_b(x)\Phi_c(x)\Phi_e(y):+(x\leftrightarrow y)]$$
$$+24g_{abcd}h'_{cd}D^F_{m^*_c m^*_d}(x-y)[:\Phi_a(x)\Phi_b(x):+(x\leftrightarrow y)]$$

$$-4h_{ab}h_{ab}D^F_{m_c}(x-y)\{: A^\mu_b(x)A_{b\mu}(y) : -[: \tilde{u}_a(x)u_b(y) : +(x \leftrightarrow y)]\}$$
$$+4h_{ab}h_{ab}D^F_{m_a m_b}(x-y)\mathbf{1} + 4h'_{ac}h'_{bc}D^F_{m_c^*}(x-y) : \Phi_a(x)\Phi_b(y) :$$
$$+4h'_{ab}h'_{ab}D^F_{m_a^* m_b^*}(x-y)\mathbf{1} \tag{3.2.49}$$

and a finite renormalization of the type $\delta(x-y)L(x)$. The finite normalization $L(x)$ is Lorentz invariant and of power $\leqslant 4$, i.e. a sum of terms of the type (3.2.17)–(3.2.19).

3.3. Second-order gauge invariance

We are not guaranteed that the generic expression of $T_2(x, y)$ from the preceding proposition leads to a well defined operator on the factor space $\mathcal{H}^{gh,r}_{\rm YM}$; as in [16], one can show that this can happen if and only if some severe restrictions are placed on the constants appearing in the expression of the interaction Lagrangian. In [3] it is proved that, in the standard model, one can choose conveniently the finite normalization $L(x)$ such that gauge invariance is valid in the second-order perturbation theory (this in turn guarantees that the factorization of the S-matrix is possible in this order). We detail below this result in a more general context, when the characteristics of the standard model are not used in the computations, i.e. we do not take specific expressions for the constants f_{abc}. As in [16] we observe that the generic expression for the second-order S-matrix obtained in the preceding proposition corresponds to a 'canonical' causal splitting of the commutator $D_2(x, y)$; namely, one splits causally the numerical distributions in the expression of the commutator by making the replacements $D_m \to D_m^{\rm ret(adv)}$, $\partial_\mu D_m \to \partial_\mu D_m^{\rm ret(adv)}$, $D_{m_a m_b;\mu\nu} \to D^{\rm ret(adv)}_{m_a m_b;\mu\nu}$, etc. In this way one obtains the expressions $R^0_2(x, y)$ and $A^0_2(x, y)$ which will be called the *canonical causal splitting*. This splitting leads to the expression $T^0_2(x, y) + T^h_2(x, y)$ from the preceding proposition. Now we have the following theorem.

Theorem 3.10. *The expression T_2 appearing in the preceding proposition leads, in the adiabatic limit, to a well defined operator on $\mathcal{H}^r_{\rm YM}$ if and only if:*

(a) the constants f_{abc} verify the Jacobi identities

$$f_{abc}f_{dec} + f_{bdc}f_{aec} + f_{dac}f_{bec} = 0, \tag{3.3.1}$$

in particular, there exists a compact Lie group G with f_{abc} as structure constants; moreover G is of the form $G \equiv H_1 \times \cdots \times H_k \times U(1) \times \cdots \times U(1)$ with $H_1, \ldots, H_k$ compact simple Lie groups;

(b) the constants f'_{abc} verify the identity

$$f'_{dca}f'_{ceb} - f'_{dcb}f'_{cea} = -f_{abc}f'_{dec}, \tag{3.3.2}$$

in other words, if we define the $r \times r$ (antisymmetric) matrices T_a, $a = 1, \ldots, r$ according to

$$(T_a)_{bc} \equiv -f'_{bca} \qquad \forall a, b, c = 1, \ldots, r \tag{3.3.3}$$

then they are an r-dimensional representation of the Lie algebra Lie(G) *determined by the structure constants f_{abc};*

(c) the constants f''_{abc} verify the following identities:

$$f'_{cab}f''_{cde} + f'_{cdb}f''_{cae} + f'_{ceb}f''_{cda} = 0 \quad \text{iff} \quad m_a = m_b = m_d = m_e = 0 \tag{3.3.4}$$

(d) the constants h_{ab} verify the identities

$$f_{abc}h_{cd} = 0 \tag{3.3.5}$$

so they can be non-null only in the Abelian sector (see (a) above);

(e) the constants h'_{ab} verify the identity

$$f'_{dba}h'_{cd} + f'_{dca}h'_{bd} = 0; \tag{3.3.6}$$

(f) the constants g_{abcd} verify the identity

$$f'_{cba} g_{cdef} + (b \leftrightarrow d) + (b \leftrightarrow e) + (b \leftrightarrow f) = 0 \tag{3.3.7}$$

for $m_a = m_b = m_d = m_e = m_f = 0$.

Proof. (i) We follow the ideas from [7] and [16]. We first have

$$d_Q D_2(x, y) = i \frac{\partial}{\partial x^\mu} [T_1^\mu(x), T_1(y)] - (x \leftrightarrow y) \tag{3.3.8}$$

and we must compute the right-hand side. It is elementary to see that the distribution $d_Q D_2(x, y)$ still has a causal support so it can be split causally:

$$d_Q D_2(x, y) = d_Q A_2(x, y) - d_Q R_2(x, y). \tag{3.3.9}$$

If we split causally the right-hand side of the formula (3.3.8) preserving Lorentz covariance and power counting, we will obviously get valid expressions for the distributions $d_Q A_2(x, y)$ and $d_Q R_2(x, y)$. If we want to obtain exactly $d_Q A_2^0(x, y)$ and $d_Q R_2^0(x, y)$ we must compute the commutators in the (3.3.8), next perform the derivatives and finally extract the canonical causal splitting. Of course, in this way we do not get the most general expression for these distributions because we have the possibility of finite normalizations. But the arbitrariness for $d_Q R_2(x, y)$ is *exactly the same* as the arbitrariness for $d_Q T_2(x, y)$, i.e. it is of the form $\delta(x - y)N(x)$. So, we get the most general expression for the distributions $d_Q A_2(x, y)$ and $d_Q R_2(x, y)$.

Because we have (3.1.12) for $n = 1$, we conclude that (3.1.12) for $n = 2$ is equivalent to

$$\int_{\mathbb{R}^4 \times \mathbb{R}^4} dx\, dy\, g_\epsilon(x) g_\epsilon(y)\, d_Q R_2(x, y) \bigg|_{\mathrm{Ker}(Q)} = O(\epsilon). \tag{3.3.10}$$

Imposing this condition on the expression determined in the way outlined above will lead to conditions (a)–(e) from the statement.

(ii) By straightforward computation we obtain the following expression for the first commutator appearing in (3.3.8):

$$[T_1^\mu(x), T_1(y)] = \frac{\partial}{\partial x_\mu} D_{m_c}(x - y) T_c(x, y) + \frac{\partial^2}{\partial x_\mu \partial x_\rho} D_{m_c}(x - y) T_{c;\rho}(x, y) \\ +(T \to T', m_c \to m_c^*) + \cdots. \tag{3.3.11}$$

The anomalies can be produced only by those terms in T_1^μ of the type $\partial^\mu A \cdots B$, and this simplifies considerably the computations. We obtain in this way only the terms from the preceding expression. Therefore we give the explicit expression of the operator-valued distributions T_c and T_c^ρ:

$$\begin{aligned} T_c(x, y) \equiv{}& f_{cab} f_{cde} [: u_a(x) A_b^\nu(x) A_d^\rho(y) F_{e\rho\nu}(y) : - : u_a(x) A_b^\nu(x) u_d(y) \partial_\nu \tilde{u}_e(y) : \\ & + \tfrac{1}{2} : u_a(x) u_b(x) A_d^\rho(y) \partial_\rho \tilde{u}_e(y) :] \\ & + f_{abc} f'_{dec} : u_a(x) A_b^\rho(x) \Phi_d(y) \partial_\rho \Phi_e(y) : \\ & + 2 f_{abc} h^{(1)}_{dec} : u_a(x) A_{b\rho}(x) \Phi_d(y) A_e^\rho(y) : \\ & - \tfrac{1}{2} f_{abc} h^{(2)}_{dec} : u_a(x) u_b(x) \Phi_d(y) \tilde{u}_e(y) : \\ & + f_{cab} h_{cd} [2 : u_a(x) A_{b\nu}(x) A_d^\nu(y) : + : u_a(x) u_b(x) \tilde{u}_d(y) :] \\ T'_c(x, y) \equiv{}& - f'_{cab} f'_{cde} : \Phi_a(x) u_b(x) \partial_\rho \Phi_d(y) A_e^\rho(y) : \\ & - f'_{cab} h^{(1)}_{cde} : \Phi_a(x) u_b(x) A_{d\rho}(y) A_e^\rho(y) : \\ & - f'_{cab} h^{(2)}_{cde} : \Phi_a(x) u_b(x) \tilde{u}_d(y) u_e(y) : \\ & - 3 f'_{cab} f''_{cde} : \Phi_a(x) u_b(x) \Phi_d(y) \Phi_e(y) : \\ & - 4 f'_{acb} g_{cdef} : \Phi_a(x) u_b(x) \Phi_d(y) \Phi_e(y) \Phi_f(y) : \\ & - 2 f'_{cab} h'_{cd} : \Phi_a(x) u_b(x) \Phi_d(y) : \end{aligned} \tag{3.3.12}$$

and

$$\begin{aligned} T_c^\rho(x,y) &\equiv -f_{cab} f_{cde} : u_a(x) A_{b\nu}(x) A_d^\nu(y) A_e^\rho(y) : \\ T_c^{'\rho}(x,y) &- f'_{cab} f'_{cde} : \Phi_a(x) u_b(x) \Phi_d(y) A_e^\rho(y) : . \end{aligned} \tag{3.3.13}$$

(iii) We must split causally the distribution $\frac{\partial}{\partial x^\mu}[T_1^\mu(x), T_1(y)]$. It is important that the causal splitting can be done in such a way that we have the relations from proposition 3.8. One can obtain in the same way the following expression for the canonical splitting:

$$\left(\frac{\partial}{\partial x^\mu}[T_1^\mu(x), T_1(y)]\right)^{\rm ret} = \frac{\partial}{\partial x_\mu}\tilde{R}_{1,\mu}(x,y) - {\rm i}\delta(x-y) A(x,y) \tag{3.3.14}$$

where

$$\tilde{R}_1^\mu(x,y) \equiv R_1^\mu(x,y) - i\delta(x-y)\sum_{c=1}^r \left[T_c^\mu(x,y) + (T_c^\mu \to T_c^{'\mu})\right] \tag{3.3.15}$$

(with $R_{1,\mu}(x,y)$ given by (3.3.11) with $D_m \to D_m^{\rm ret}$, etc) and the anomaly is given by

$$A(x,y) \equiv \sum_{c=1}^r \left[T_c(x,y) - \frac{\partial}{\partial x^\rho} T_c^\rho(x,y) + (T \to T')\right]. \tag{3.3.16}$$

The factorization condition (3.3.10) can now be written as follows:

$$\int_{\mathbb{R}^4} {\rm d}x\, (g_\epsilon(x))^2 [2A(x) - {\rm d}_Q L(x)]_{{\rm Ker}(Q)} = {\rm O}(\epsilon) \tag{3.3.17}$$

where $L(x)$ is a finite normalization and $A(x) \equiv A(x,x)$.

After performing the computations and with some rearrangement, the last condition is as follows:

$$\begin{aligned} \int_{\mathbb{R}^4} {\rm d}x\, g_\epsilon(x)^2 [(2f_{cae} f_{cdb} - f_{cab} f_{cde}) &: u_a(x) F_{b\rho\nu}(x) A_d^\nu(x) A_e^\rho(x) : \\ & - f_{cab} f_{cbe} : \partial_\rho u_a(x) A_{b\nu}(x) A_d^\rho(x) A_e^\nu(x) : \\ & + (2f_{cad} f_{cbe} - f_{cab} f_{cde}) : u_a(x) u_b(x) A_d^\rho(x) \partial_\rho \tilde{u}_e(x) : \\ & + 2(f_{abc} f'_{dec} - 2f'_{cda} f'_{ceb}) : u_a(x) A_b^\rho(x) \Phi_d(y) \partial_\rho \Phi_e(y) : \\ & + 2(2f_{abc} h^{(1)}_{dec} - f'_{cda} h^{(1)}_{ceb}) : u_a(x) A_{b\rho}(x) \Phi_d(y) A_e^\rho(y) : \\ & - (f_{abc} h^{(2)}_{dec} - 2f'_{cda} h^{(2)}_{ceb}) : u_a(x) u_b(x) \Phi_d(y) \tilde{u}_e(y) : \\ & - 2f'_{cab} f'_{cde} : \Phi_a(x) u_b(x) \Phi_d(y) \partial_\rho A_e^\rho(y) : \\ & - 6f'_{cab} f''_{cde} : \Phi_a(x) u_b(x) \Phi_d(y) \Phi_e(y) : \\ & + 4f'_{cba} g_{cdef} : u_a(x) \Phi_b(x) \Phi_d(x) \Phi_e(x) \Phi_f(x) : \\ & - 4f_{cab} h_{cd} : u_a(x) A_{b\nu}(x) A_d^\nu(x) : -2f_{cab} h_{cd} : u_a(x) u_b(x) \tilde{u}_d(x) : \\ & - 4f'_{cab} h'_{cd} : \Phi_a(x) u_b(x) \Phi_d(x) : - {\rm d}_Q L(x)]|_{{\rm Ker}(Q)} = {\rm O}(\epsilon). \end{aligned} \tag{3.3.18}$$

One has to compute the expression ${\rm d}_Q L(x)$ taking into account the generic form for $L(x)$ described in the preceding subsection. One takes into account (3.2.17)–(3.2.19) and the corresponding expressions from [16]; to avoid confusion we will append a tilde sign to all coefficients in these expressions. We equate with zero the coefficients of the linearly independent (integrated) Wick monomials. We now have the following cases:

- We consider the coefficients of the (linearly independent) integrated Wick monomials $\int_{\mathbb{R}^4} {\rm d}x\, g_\epsilon^2 \colon u_a u_b A_d^\rho \partial_\rho \tilde{u}_e \colon$ $\int_{\mathbb{R}^4} {\rm d}x\, g_\epsilon^2 \colon \partial_\rho u_a u_b A_d^\rho \tilde{u}_e \colon$ $\int_{\mathbb{R}^4} {\rm d}x\, g_\epsilon^2 \colon A_{a\mu} A_b^\mu u_c \partial_\rho A_d^\rho \colon$ and $\int_{\mathbb{R}^4} {\rm d}x\, g_\epsilon^2 \colon u_a A_b^\mu A_{d\nu} A_e^\nu \colon$ and we get, as in [16], that the constants f_{abc} verify the Jacobi identity (3.3.1) so we have (a) from the statement. Moreover we obtain the explicit expression of $\tilde{g}^{(1)}_{abde}$

$$\tilde{g}^{(1)}_{abde} = \frac{\mathrm{i}}{8}(f_{cad} f_{cbe} + f_{cae} f_{cbd}). \tag{3.3.19}$$

- From the coefficients of the Wick monomials $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon u_a u_b \tilde{u}_d\colon$ $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon \Phi_a \partial_\mu A^\mu_b u_d\colon$ $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon \Phi_a \partial^\mu \Phi_b \partial_\mu u_d\colon$ $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon \colon \partial_\nu A_{a\mu} \partial^\mu A^\nu_b u_d\colon$ $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon u_a \partial_\mu A_{b\nu} \partial^\mu A^\nu_d\colon$ and $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon u_a \partial_\mu A^\mu_b \partial_\nu A^\nu_d\colon$ we obtain (3.3.5).
- From the coefficients of the monomials $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon u_a A^\rho_b \Phi_d \partial_\rho \Phi_e\colon$ $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon u_a A_{b\rho} \Phi_d A^\rho_e\colon$ $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon u_a u_b \Phi_d \tilde{u}_e\colon$ $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon \Phi_a u_b \Phi_d \Phi_e\colon$ and $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon \Phi_a \Phi_b \partial_\mu A^\mu_d u_e\colon$ we obtain the following system of equations which is harder to analyse than the previous ones:

$$2(f_{abc} f'_{dec} - 2 f'_{cda} f'_{ceb}) + 4\mathrm{i}\tilde{g}^{(5)}_{deba} = 0 \tag{3.3.20}$$

$$[(2 f_{abc} h^{(1)}_{dec} - f'_{cda} h^{(1)}_{ceb}) + (b \leftrightarrow e)] - 2\mathrm{i}\tilde{g}^{(5)}_{adbe} m_a = 0 \tag{3.3.21}$$

$$(f_{abc} h^{(2)}_{dec} - 2 f'_{cda} h^{(2)}_{ceb}) - (a \leftrightarrow b)] + \mathrm{i}[\tilde{g}^{(6)}_{adeb} m_a - (a \leftrightarrow b)] = 0 \tag{3.3.22}$$

$$6 f'_{cab} f''_{cde} - \mathrm{i}\tilde{g}^{(6)}_{deab} + 4\mathrm{i}\tilde{g}^{(7)}_{abde} m_b + \text{cyclic perm } (a, d, e) = 0 \tag{3.3.23}$$

$$-(f'_{cae} f'_{cbd} + f'_{cbe} f'_{cad}) + 2\mathrm{i}\tilde{g}^{(5)}_{abde} + \mathrm{i}\tilde{g}^{(6)}_{abde} = 0. \tag{3.3.24}$$

It is very convenient that this system can be solved explicitly. First, we take into account that the constants $\tilde{g}^{(5)}_{deba}$ are symmetric in d and e and also in b and a. So, if we take the antisymmetric part in d and e of the relation (3.3.20), we get the relation (3.3.2) from the statement, and from the symmetric part we obtain the explicit expression for $\tilde{g}^{(5)}_{deba}$:

$$\tilde{g}^{(5)}_{deba} = -\frac{\mathrm{i}}{2}(f_{cda} f'_{ceb} + f'_{cdb} f'_{cea}). \tag{3.3.25}$$

If we substitute this expression into equation (3.3.24) we get

$$\tilde{g}^{(6)}_{abde} = 0. \tag{3.3.26}$$

Then, relation (3.3.22) becomes a consequence of (3.3.2) if we use the explicit expression (3.2.36) for $h^{(2)}_{abc}$.

Next, we introduce the expressions (3.3.25) and (3.2.35) of $\tilde{g}^{(5)}_{abde}$ and resp. $h^{(1)}_{cab}$ into equation (3.3.21) and we obtain an identity if we take into account that the constants f'_{abc} verify equations (3.2.12) and (3.3.2).

If we now substitute (3.3.26) into equation (3.3.23) we obtain easily the condition (3.3.4) from the statement; moreover we obtain the explicit expression for $\tilde{g}^{(7)}_{abde}$:

$$\tilde{g}^{(7)}_{abde} = \frac{\mathrm{i}}{2m_b}(f'_{cab} f''_{cde} + \text{cyclic perm } a, d, e). \tag{3.3.27}$$

- If we consider now the coefficient of the Wick monomial $\int_{\mathbb{R}^4} \mathrm{d}x\, g^2_\epsilon\colon u_a \Phi_b \Phi_c\colon$ we get

$$3\mathrm{i}\tilde{h}^{(3)}_{abc} m_a + 2(f'_{dba} h'_{cd} + f'_{dca} h'_{bd}) = 0. \tag{3.3.28}$$

For $m_a = 0$ we obtain condition (3.3.6) from the statement. We also get

$$\tilde{h}^{(3)}_{abc} = \frac{2\mathrm{i}}{3m_a}(f'_{dba} h^{(3)}_{cd} + f'_{dca} h^{(3)}_{bd}) \qquad \text{for} \quad m_a \neq 0. \tag{3.3.29}$$

- Finally, we consider the integrated Wick monomial $\int_{\mathbb{R}^4} \mathrm{d}x\, g_{abdef}\colon u_a \Phi_b \Phi_d \Phi_e \Phi_f\colon$ where $m_d = m_e = m_f = 0$. In this case we get an expression for $\tilde{g}^{(8)}$:

$$\tilde{g}^{(8)}_{abdef} \equiv -\frac{i}{5m_a}(f'_{cba} g_{cdef} + f'_{cda} g_{cbef} + f'_{cea} g_{cbdf} + f'_{cfa} g_{cbde}) \tag{3.3.30}$$

and obtain (f) from the statement.

We have obtained all the relations from the statement and it is clear that we have used all aspects of equation (3.3.18). □

Remark 3.11. The representation T_a exhibited in the statement of the theorem is nothing else but the representation of the gauge group G into which the Higgs fields live.

Remark 3.12. Much of the effort from the appendix of [3] is nothing more than the painful verification that the standard model fulfils conditions (3.2.12) and (3.3.2), and that all other equations are identically verified. The advantage of our approach consists in exhibiting very clearly where the computational difficulties are hidden.

To verify condition (3.3.2) in specific models it is convenient to detail this relation. We have by an elementary analysis the following.

Corollary 3.13. *The relation (3.3.2) is equivalent to the following set of relations:*

$$\sum_{m_c\neq 0}(f_{abc}g_{dec}+f_{ebc}f_{dac})=0 \qquad \text{for} \quad m_b=m_d=0 \qquad m_a\neq 0 \qquad m_e\neq 0 \tag{3.3.31}$$

$$2\sum_{m_c\neq 0} f_{abc}g_{dec}m_c m_e+\sum_{m_c\neq 0}\frac{1}{m_c}[f_{ceb}g_{dca}m_a(m_c^2+m_e^2-m_b^2)-(a\leftrightarrow b)] + \sum_{m_c=0}[f'_{dca}g_{ceb}m_b-(a\leftrightarrow b)]=0 \qquad \text{for} \quad m_d=0 \qquad m_e\neq 0 \qquad m_a\neq 0 \qquad m_b\neq 0 \tag{3.3.32}$$

$$2\sum_{m_c\neq 0}(m_d^2+m_e^2-m_c^2)f_{abc}f_{dec} - \sum_{m_c\neq 0}\frac{1}{m_c^2}[f_{cda}f_{ceb}(m_c^2+m_d^2-m_a^2)(m_c^2+m_e^2-m_b^2) -(a\leftrightarrow b)]-4m_am_bm_dm_e \times\sum_{m_c=0}[g_{dca}g_{ceb}-(a\leftrightarrow b)]=0 \qquad \text{for} \quad m_d\neq 0 \qquad m_e\neq 0 \tag{3.3.33}$$

$$\sum_{m_c\neq 0} f_{abc}f'_{dec}-m_am_b\sum_{m_c\neq 0}(g_{dca}g_{ceb}-(a\leftrightarrow b)) - \sum_{m_c=0}(f'_{cda}f'_{ceb}-(a\leftrightarrow b))=0 \qquad \text{for} \quad m_d=m_e=0. \tag{3.3.34}$$

Proof. One considers the separately distinct cases of (3.3.2), namely when m_d and m_e are both equal to 0, both non-null, or only one of them is equal to 0 and one obtains, respectively, (3.3.34), (3.3.33) and (3.3.31), (3.3.32) if one substitutes the explicit expressions (3.2.37) and (3.2.38) for f'_{abc}. □

Now we have, as in [16], corollary 4.8.

Corollary 3.14. *Suppose that the constants f_{abc}, f_{abc}, h_{ab} and h'_{ab} verify the conditions from the statements of theorems 3.1 and 3.10. Then, the general expression for the chronological product T_2 is given by the sum of the particular solution:*

$$T_2^c(x,y) =: T_1(x)T_1(y):+T_2^0(x,y)+T_2^h(x,y)+i\delta(x-y) \times[\tfrac{1}{4}f_{cab}f_{cde}:A_{a\nu}(x)A_{b\nu}(x)A_d^{\mu}(x)A_e^{\nu}(x): -f'_{cda}f'_{ceb}:A_{a\nu}(x)A_b^{\nu}(x)\Phi_d(x)\Phi_e(x):$$

$$+ \sum_{m_a,m_b,m_d,m_e\neq 0} \frac{3}{2m_b} f'_{cab} f''_{cde} : \Phi_a(x)\Phi_b(x)\Phi_d(x)\Phi_e(x) :$$
$$- \sum_{m_a\neq 0} \frac{4}{5m_a} f'_{cba} g_{cdef} : \Phi_a(x)\Phi_b(x)\Phi_d(x)\Phi_e(x)\Phi_f(x) :$$
$$- \sum_{m_a\neq 0} \frac{2}{3m_a} f'_{dba} h'_{cd} : \Phi_a(x)\Phi_b(x)\Phi_c(x) : +L(x)] \tag{3.3.35}$$

and a finite renormalization of the type $\mathrm{i}\delta(x-y)N(x)$. *Here the expressions* $T_2^0(x,y)$ *and* $T_2^h(x,y)$ *have been defined previously (see the formulae (3.2.48), (3.2.49)) and the Wick monomial* $N(x)$ *is a finite normalization of the type (3.2.2). In particular, the theory is renormalizable up to order two. The condition of unitarity can be satisfied if and only if* $N(x)^\dagger = N(x)$.

We only note that the expression of the finite normalization follows from expressions (3.3.19), (3.3.25)–(3.3.27) of $\tilde{g}^{(1)}_{abcd}$ and $\tilde{g}^{(5)}_{abcd} - \tilde{g}^{(8)}_{abcd}$ and $\tilde{h}^{(3)}_{abc}$.

Remark 3.15. It was noticed in [7] that the expression T_{11} from theorem 3.1 and the first finite normalization from the preceding formula reconstruct the usual Yang–Mills Lagrangian. A similar remark is in order in this context, namely the expression T_{12} from theorem 3.1 and the second finite normalization from the preceding formula reconstruct the usual kinematic part of the Higgs Lagrangian (see, for instance, [29]). The third normalization is a part of the Higgs potential.

3.4. The standard model

It was clear from the preceding sections that in order to specify a certain concrete model of heavy spin-1 bosons it is not sufficient to specify the gauge group G from theorem 3.1 but one also needs to fix a basis in the Lie algebra $\mathrm{Lie}(G)$. This is a consequence of the fact that the assignment of the masses m_a, m_b, etc is connected with a specific basis, and if we choose another basis we will obtain fields which do not create particles of fixed mass.

For the case of the standard model it means that we have to specify the group, which in this case is $SU(2)\times U(1)$ *and* the basis through the *Weinberg angle*. Explicitly, let us take in the Lie algebra of $SU(2)\times U(1)$ the standard basis X_a, $a = 0, 1, 2, 3$ with the commutation relations

$$[X_a, X_b] = \epsilon_{abc} X_c \qquad a,b = 1,2,3 \qquad [X_0, X_a] = 0 \qquad a = 1,2,3. \tag{3.4.1}$$

We consider another basis Y_a, $a = 0, 1, 2, 3$ defined by

$$\begin{aligned} &Y_a = gX_a \qquad a = 1,2 \qquad Y_3 = -g\cos\theta X_3 + g'\sin\theta X_0 \\ &Y_0 = -g\sin\theta X_3 - g'\cos\theta X_0. \end{aligned} \tag{3.4.2}$$

By definition, the angle θ, determined by the condition $\cos\theta > 0$ is called the *Weinberg angle*. The constants g and g' are real with $g > 0$. Then one can show that the new commutation rules produce the following structure constants [3]:

$$f_{210} = g\sin\theta \qquad f_{321} = g\cos\theta \qquad f_{310} = 0 \qquad f_{320} = 0 \tag{3.4.3}$$

and the rest of the constants are determined by antisymmetry. By definition, the *standard model* corresponds to this choice of constants *and* to the following assignment of masses:

$$m_0 = 0 \qquad m_a \neq 0 \qquad a = 1,2,3. \tag{3.4.4}$$

We say that the particles created by A_0^μ are *photons* and the particles created by A_a^μ, $a = 1, 2, 3$ are *heavy Bosons* (more precisely, for $a = 1, 2$ we have the *W-Bosons* and for $a = 3$ the *Z-Boson*).

We will derive below, directly from our general analysis, that the standard model is compatible with all restrictions outlined in the previous analysis and we will see that the only free parameters are, essentially, m_0^H and f''_{000}.

Theorem 3.16. *In the standard model, the following relations are true:*

(a) the masses of the heavy bosons are constrained by

$$m_1 = m_2 = m_3 \cos\theta; \tag{3.4.5}$$

(b) the constants f'_{abc} are completely determined by the antisymmetry property (3.2.10) and

$$\begin{aligned} f'_{011} &= f'_{022} = \frac{\epsilon g}{2} \qquad f'_{033} = \frac{\epsilon g}{2\cos\theta} \qquad f'_{210} = g\sin\theta \\ f'_{321} &= -f'_{312} = \frac{g}{2} \qquad f'_{123} = -g\frac{\cos 2\theta}{2\cos\theta} \end{aligned} \tag{3.4.6}$$

the rest of them being zero. Here ϵ can take the values + or −;

(c) the constants f''_{abc} are (partially) determined by

$$\begin{aligned} f''_{abc} &= 0 \qquad \text{for} \quad a,b,c = 1,2,3 \qquad f''_{00a} = 0 \qquad \text{for} \quad a \neq 0 \\ f''_{0ab} &= 0 \qquad a = 1,2,3 \qquad a \neq b \qquad f''_{0aa} = \frac{\epsilon g}{12 m_1}(m_0^H)^2 \qquad a = 1,2,3. \end{aligned} \tag{3.4.7}$$

Proof. (i) We first consider the consistency relation (3.2.9) and immediately get that $m_1 = m_2$. The consistency condition (3.2.11) is trivial because from the antisymmetry property (3.2.10) we have

$$f'_{00a} = 0 \qquad a = 0, 1, 2, 3. \tag{3.4.8}$$

(ii) We investigate now the consistency condition (3.3.2). It is convenient to use it in the detailed form (3.3.31)–(3.3.34). We mention briefly here the result of elementary computations. From (3.3.31) we obtain equivalently that

$$g_{0ab} = 0 \qquad a, b = 1, 2, 3 \qquad a \neq b \qquad g_{011} = g_{022}. \tag{3.4.9}$$

From (3.3.32) only the case $a = 1, b = 2$ gives something non-trivial, namely

$$g_{033} = g_{011} = g_{022}. \tag{3.4.10}$$

Next, we consider relation (3.3.33). From the case $d = 1, e = 2, a = 1, b = 2$ we get

$$m_1^2 g_{011}^2 = g^2\left(1 - \frac{3m_3^2}{4m_1^2}\cos^2\theta\right) \tag{3.4.11}$$

and from the case $d = 1, e = 2, a = 2, b = 3$

$$g_{022}g_{033} = g^2\frac{m_3^2}{4m_1^4}\cos^2\theta, \tag{3.4.12}$$

all other cases give identities. We observe now that the last two relations are consistent iff we have

$$m_1 = m_3\cos\theta. \tag{3.4.13}$$

Finally, relation (3.3.34) is trivial. From the preceding relations, we can reconstruct all the constants f'_{abc} as given in the statement.

(iii) Now we consider the relation (3.3.4). It is not very hard to prove that if we also take into account (3.2.13) we obtain only relations (c) from the statement.

(iv) The relation (3.3.6) gives

$$h_{ab} = 0 \qquad a \neq b \qquad h_{11} = h_{22} = h_{33} = 2h'_{00}. \tag{3.4.14}$$

(v) Finally, the conditions (3.3.7) give no restrictions. We have obtained all the relations from the statement. □

Remark 3.17. In the standard model one disregards the terms T_{15} and T_{16} from theorem 3.1 and it follows that the expression T_2^h can be put to zero. One can also show [1] that it is possible to fix $\epsilon = +$. We note that we are left only with the Yang–Mills interaction of the usual form. However, now we can carry out all the computations completely rigorously, after we have conveniently split the distributions involved in the analysis. The choice $g = \frac{e}{\sin\theta}, \quad g' = -\frac{e}{\cos\theta}$ can be obtained if one includes interaction with matter and requires that the interaction of the electron Dirac field with the electromagnetic potential has the usual form.

We note in the end two facts. First, because of the equality $m_1 = m_2$ there exists a global symmetry of the theory, namely the *electric charge* which commutes with the S-matrix.

Next, suppose we admit that the photon has a small non-zero mass $m_0 \neq 0$ and we try to interpret the adiabatic limit as the process $\lim_{m_0 \searrow 0} \lim_{\epsilon \searrow 0}$. One can easily prove that this is not possible. Indeed, if all the masses m_a, $a = 0, \ldots, 3$ are non-zero, then the expressions f'_{abc} are given by the expressions (3.2.37) for all values of the indices. One can plug this expression into the relation (3.3.2), if one considers, for instance, the cases $a = d = 0$, $b = e = 1$ and obtains the following relation:

$$m_0^4 + 2(m_2^2 - m_1^2)m_0^2 + (m_1^2 - m_2^2)(m_1^2 + 3m_2^2) = 0. \tag{3.4.15}$$

In the case $a = d = 0$, $b = e = 2$ one obtains the preceding relation with $m_1 \leftrightarrow m_2$. If we subtract the two relations, then we get

$$(m_1^2 - m_2^2)(m_0^2 - m_1^2 - m_2^2) = 0. \tag{3.4.16}$$

We have two cases: if $m_1 = m_2$ then from (3.4.15) we obtain $m_0 = 0$; if $m_0^2 = m_1^2 + m_2^2$ then again (3.4.15) gives $m_1 m_2 = 0$. So, we obtain that at least one of the masses m_a, $a = 0, 1, 2$ must be null, which contradicts the hypothesis that all masses are non-zero.

4. Conclusions

We have analysed in full the possibilities of coupling non-trivially heavy bosons of spin-1 up to order two of the perturbation theory. In particular we have reobtained in a rather elementary way the standard model (without leptons). In a subsequent publication [17] we will investigate, in our more general framework, the case when the leptons are included. In particular it is expected, according to the usual analysis (see also [3]) that, going to the third order of the perturbation theory, we will find out new restrictions on the parameters f''_{000} and g_{0000} and also some restrictions on the Fermion sector, namely the cancellation of some anomaly (of the Adler–Bell–Jackiw type).

Another extremely interesting problem is to investigate the class of Lie groups for which there exists a non-trivial solution to our problem. Indeed, it is not obvious that any Lie group of the type described in the statement of theorem 3.1 admits a representation of dimension equal to the dimension of the group, realized by antisymmetric matrices and verifying the mass relation (3.2.12). In the absence of a general solution, one should test the existence of a non-trivial solution of the perturbation series, at least for a simple Lie group like $SU(5)$, because such groups are characteristic of grand unified theories.

Finally, one should find explicit expressions for the distributions of the type $D^F_{m_a m_b \cdots}$ and perform rigorous computations for various cross sections of the standard model. In this way one could check if some differences with respect to the usual computational approaches to the standard model appear or, more probably, prove that one obtains the same results.

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